Higher Anomalies, Higher Symmetries, and Cobordisms I:

Classification of Higher-Symmetry-Protected Topological States and Their Boundary Fermionic/Bosonic Anomalies via a Generalized Cobordism Theory

Zheyan Wan[1,2] and Juven Wang[3,4]

[1] *Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China*

[2] *School of Mathematical Sciences, USTC, Hefei 230026, China*

[3] *School of Natural Sciences, Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA*

[4] *Center of Mathematical Sciences and Applications, Harvard University, Cambridge, MA 02138, USA*

Abstract

By developing a generalized cobordism theory, we explore the higher global symmetries and higher anomalies of quantum field theories and interacting fermionic/bosonic systems in condensed matter. Our essential math input is a generalization of Thom-Madsen-Tillmann spectra, Adams spectral sequence, and Freed-Hopkins's theorem, to incorporate higher-groups and higher classifying spaces. We provide many examples of bordism groups with a generic H-structure manifold with a higher-group $\mathbb{G}$, and their bordism invariants — e.g. perturbative anomalies of chiral fermions [originated from Adler-Bell-Jackiw] or bosons with U(1) symmetry in any even spacetime dimensions; non-perturbative global anomalies such as Witten anomaly and the new SU(2) anomaly in 4d and 5d. Suitable H such as SO/Spin/O/Pin$^\pm$ enables the study of quantum vacua of general bosonic or fermionic systems with time-reversal or reflection symmetry on (un)orientable spacetime. Higher 't Hooft anomalies of dd live on the boundary of $(d+1)$d higher-Symmetry-Protected Topological states (SPTs) or symmetric invertible topological orders (i.e., invertible topological quantum field theories at low energy); thus our cobordism theory also classifies and characterizes higher-SPTs. Examples of higher-SPT's anomalous boundary theories include strongly coupled non-Abelian Yang-Mills (YM) gauge theories and sigma models, complementary to physics obtained in [arXiv:1810.00844, 1812.11955, 1812.11968, 1904.00994].

This article is a companion with further detailed calculations supporting other shorter articles.

e-mail: wanzy@mail.ustc.edu.cn (Corresponding Author)

e-mail: juven@ias.edu (Corresponding Author)

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1 Introduction and Summary

1.1 Preliminaries

Thom, as the pioneer of bordism theory, studied the criteria when the disjoint union of two closed n-manifolds is the boundary of a compact $(n+1)$-manifold [1]. Thom found that this relation is an equivalence relation on the set of closed n-manifolds. Moreover, the disjoint union operation defines an abelian group structure on the set of equivalence classes. This group is called the unoriented bordism group, it is denoted by Ω_n^{O}. Furthermore, Thom found that the Cartesian product defines a graded ring structure on $\Omega_*^{\mathrm{O}} := \bigoplus_{n\geq 0} \Omega_n^{\mathrm{O}}$, which is called the unoriented bordism ring. Thom also found that the bordism invariants of Ω_n^{O} are the Stiefel-Whitney numbers. Namely, two manifolds are unorientedly bordant if and only if they have identical sets of Stiefel-Whitney characteristic numbers. This yields many interesting consequences. For example, the real projective space $\mathbb{RP}^2$ is not a boundary while $\mathbb{RP}^3$ is; also the complex projective space $\mathbb{CP}^2$ and $\mathbb{RP}^2 \times \mathbb{RP}^2$ are unorientedly bordant.

Many generalizations are made to bordism theories so far. For example, we can consider manifolds which are equipped with an H-structure, we follow the definition of H-structure given in [2]. Our work is inspired by a cobordism theory from the Madsen-Tillmann spectrum [3] and from Freed-Hopkins [4]. Freed and Hopkins propose a cobordism theory [4] to classify the Symmetry-Protected Topological states (SPTs) [5] in condensed matter physics [6] with ordinary internal global symmetries of group G and their classifying space $\mathrm{B}G$. Examples of SPTs include the famous topological insulators and topological superconductors [7, 8].

The major motivation of our work is to generalize the calculations and the cobordism theory of Freed-Hopkins [4] — such that, instead of the ordinary group G or ordinary classifying space $\mathrm{B}G$, we consider a generalized cobordism theory studying manifolds (i.e., spacetime manifolds) endorsed with H structure, with additional higher group $\mathbb{G}$ (i.e., generalized as principal-$\mathbb{G}$ bundles) and *higher classifying spaces* $\mathrm{B}\mathbb{G}$. We consider this particular generalized cobordism theory in order to study, characterize and classify:

1. Generalized higher global symmetries of $\mathbb{G}$ [9] in physics [1] and their higher classifying spaces $\mathrm{B}\mathbb{G}$.

[1] Generalized higher global symmetries may or may not be higher-differential form global symmetries. For example, there exist certain fermionic SPTs whose higher global symmetries whose charged objects are not in terms of higher-differential forms, see Ref. [10] and References therein.

2. Higher-Symmetry-Protected Topological states (higher-SPTs), which are nontrivial quantum vacua protected by higher global symmetries of $\mathbb{G}$. Higher SPTs are characterized by (co)bordism invariants obtained from (co)bordism groups of higher classifying spaces $\mathrm{B}\mathbb{G}$. For example, in $(d+1)$-dimensional spacetime, denoted $(d+1)$d, consider the quantum vacua of internal global symmetry $\mathbb{G}$ on a $(d+1)$d spacetime manifold with H-structure, we will propose a bordism group $\Omega^H_{(d+1)}(\mathrm{B}\mathbb{G})$ and a related cobordism group $\mathrm{TP}_{(d+1)}(H \times \mathbb{G})$ to classify higher-SPTs in $(d+1)$d. See the earlier pioneer work on higher-SPTs in [11, 12].

3. Higher quantum anomalies, e.g. higher 't Hooft anomalies: The ordinary 't Hooft anomalies [13] of global symmetry G is the anomaly for QFT of the ordinary global symmetry G. In comparison, given the internal higher global symmetry $\mathbb{G}$ and the dd spacetime manifold with H-structure, we can ask what are the possible higher quantum anomalies in the dd physical theories? The associated higher anomalies, given by the data $\mathbb{G}$ and H, in the dd spacetime, via a generalization of the anomaly-inflow picture [14], turns out to relate to the anomalies of the dd boundary theory (called the boundary anomalies) of $(d+1)$d higher-SPTs (given by the same data $\mathbb{G}$ and H). So the characterization and classification of $(d+1)$d higher-SPTs in the previous remark turns out to help on the characterization and classification of dd higher 't Hooft anomalies.

   Modern examples of higher 't Hooft anomalies are found in quantum field theories (QFTs) including Yang-Mills gauge theories [15] and sigma models. The first example of higher 't Hooft anomalies is discovered by a remarkable work Ref. [16] for a pure 4d SU(N) Yang-MiIls gauge theory of even integer N with a second Chern-class topological term (called the θ-term or $\theta\mathrm{Tr}[F \wedge F]$-term in particle physics.) Further new higher 't Hooft anomalies are found in [17–20].

In summary, as we have said, we aim to study higher global symmetries, characterize and classify higher-SPTs and higher quantum anomalies.

- By *characterization*, we mean that given certain physics phenomena or theories (here, higher-SPTs and higher quantum anomalies), we like to write down their mathematical invariants (here, we mean the bordism invariant) to fully describe or capture their essences/properties. Hopefully, we can further compute their physical observables.

- By *classification*, we mean that given the spacetime dimensions (here $d+1$d for higher-SPTs or dd for higher quantum anomalies), their H-structure and the internal higher global symmetry $\mathbb{G}$, we aim to know how *many classes* (a number to count them) there are? Also, we aim to determine the mathematical structures of classes (i.e. here group structure as for (co)bordism groups: would the classes be a finite group $\mathbb{Z}_n$ or an infinite group $\mathbb{Z}$ or their mixing, etc.).

Another purpose of this article is a companion article with further detailed mathematical calculations in order to support other shorter articles [18–20].

In this Introduction, we will provide some basic physics preliminaries in Sec. 1.2 and mathematical preliminaries in Sec. 1.3. Since the concepts of higher symmetries and higher anomalies are crucial, we will also clarify what precisely we mean by higher symmetries/anomalies in condensed matter, in QFTs and in mathematics, in Sec. 1.4.

After some additional introduction to mathematical background in Sec. 2, we will provide

explicit interpretations of familiar examples (to QFT-ist and physicists) of *perturbative* anomalies in Sec. 3.1:

(1): Perturbative fermionic anomalies from chiral fermions with U(1) symmetry, originated from Adler-Bell-Jackiw (ABJ) anomalies [21,22].

(2): Perturbative bosonic anomalies from bosonic systems with U(1) symmetry.

We will also provide more exotic *non-perturbative global* anomalies in Sec. 3.2:

(3): The SU(2) anomaly of Witten [23].

(4): A new SU(2) anomaly [24],

matching the physics results of dd anomalies to mathematical cobordism group calculations in $(d+1)$d.

We briefly comment the difference between a previous cobordism theory [4,25] and this work: In all Adams charts of the computation in [4,25], there are no nonzero differentials, while in this paper we encounter nonzero differentials d_n due to the (p,p^n)-Bockstein homomorphisms in the computation involving $\mathrm{B}^2\mathbb{Z}_{p^n}$ and $\mathrm{B}\mathbb{Z}_{p^n}$.

1.2 Physics preliminaries

Freed-Hopkins's work [4] is motivated by the development of cobordism theory classification [26,27] of so-called the Symmetry Protected Topological (SPT) state in condensed matter physics [6]. In a very short summary, Freed-Hopkins's work [4] applies the theory of Thom-Madsen-Tillmann spectra [1,3], to prove a theorem relating the "Topological Phases" (which later will be abbreviated as TP) or certain deformation classes of reflection positive invertible n-dimensional extended topological field theories (iTQFT) with symmetry group (or in short, symmetric iTQFT), to Madsen-Tillmann spectrum [3] of the symmetry group.

Here an n-dimensional extended topological field theory is a symmetric monoidal functor F from the (∞,n)-category of extended cobordisms $\mathrm{Bord}_n(H_n)$ to a symmetric monoidal (∞,n)-category $\mathcal{C}$ where $\mathrm{Bord}_n(H_n)$ is defined as follows (all manifolds are equipped with H-structures, see definition 1):

- objects are 0-manifolds;
- 1-morphisms are 1-cobordisms between objects;
- 2-morphisms are 2-cobordisms between 1-morphisms;
- ...
- n-morphisms are n-cobordisms between $(n-1)$-morphisms;

- $(n+1)$-morphisms are diffeomorphisms between n-morphisms;
- $(n+2)$-morphisms are smooth homotopies between $(n+1)$-morphisms;
- ...

An n-dimensional extended topological field theory is called invertible if F factor through the Picard groupoid $\mathcal{C}^\times$. By a theorem of Galatius-Madsen-Tillmann-Weiss [3], the classifying space of $\mathrm{Bord}_n(H_n)$ is exactly the 0-th space of the Madsen-Tillmann spectrum $\Sigma^n MTH_n$.

In this work, we will consider the generalization of [4] to include higher symmetries [9], for example, including both 0-form symmetry of group $G_{(0)}$ and 1-form symmetry of group $G_{(1)}$, or in certain cases, as higher symmetry group of higher n-group.[2] Other physics motivations to study higher group can be found in [28–31] and references therein.

We generalize the work of Freed-Hopkins [4]: there is a 1:1 correspondence

$$\left\{\begin{array}{c}\text{deformation classes of reflection positive}\\ \text{invertible } n\text{-dimensional extended topological}\\ \text{field theories with a symmetry group } H_n\times\mathbb{G}\end{array}\right\}\cong[MT(H\times\mathbb{G}),\Sigma^{n+1}I\mathbb{Z}]_{\text{tors}}. \tag{1.1}$$

where H is the space time symmetry, $\mathbb{G}$ is the internal symmetry which is possibly a higher group, $MT(H\times\mathbb{G})$ is the Madsen-Tillmann spectrum [3] of the group $H\times\mathbb{G}$, Σ is the suspension, $I\mathbb{Z}$ is the Anderson dual spectrum, and tors means the torsion part.

Since there is an exact sequence

$$0\to\mathrm{Ext}^1(\pi_n\mathcal{B},\mathbb{Z})\to[\mathcal{B},\Sigma^{n+1}I\mathbb{Z}]\to\mathrm{Hom}(\pi_{n+1}\mathcal{B},\mathbb{Z})\to 0 \tag{1.2}$$

for any spectrum $\mathcal{B}$, especially for $MT(H\times\mathbb{G})$. The torsion part $[MT(H\times\mathbb{G}),\Sigma^{n+1}I\mathbb{Z}]_{\text{tors}}$ is $\mathrm{Ext}^1((\pi_n MT(H\times\mathbb{G}))_{\text{tors}},\mathbb{Z})=\mathrm{Hom}((\pi_n MT(H\times\mathbb{G}))_{\text{tors}},\mathrm{U}(1))$.

By the generalized Pontryagin-Thom isomorphism (1.10), $\pi_n MT(H\times\mathbb{G})=\Omega_n^{H\times\mathbb{G}}=\Omega_n^H(\mathrm{B}\mathbb{G})$ which is the bordism group defined in definition 2.

Namely, we can classify the deformation classes of symmetric iTQFTs and also symmetric invertible topological orders (iTOs), via the particular group

$$\mathrm{TP}_n(H\times\mathbb{G})\equiv[MT(H\times\mathbb{G}),\Sigma^{n+1}I\mathbb{Z}]. \tag{1.3}$$

Here TP means the abbreviation of "Topological Phases" classifying the above symmetric iTQFT, the torsion part of $\mathrm{TP}_n(H\times\mathbb{G})$ and $\Omega_n^H(\mathrm{B}\mathbb{G})$ are the same.

[2] For the physics application of our result, please see [18–20]. Some of these 4d non-Abelian SU(N) Yang-Mills [15]-like gauge theories can be obtained from gauging the time-reversal symmetric SU(N)-SPT generalization of topological insulator/superconductor (TI/SC) [25]. We can understand their anomalies of 0-form symmetry of group $G_{(0)}$ and 1-form symmetry of group $G_{(1)}$, as the obstruction to regularize the global symmetries locally in its own dimensions (4d for YM theory). Instead, in order to regularize the global symmetries locally and onsite, the 4d gauge theories need to be placed on the boundary of 5d higher SPTs. The 5d higher SPTs corresponds to the nontrivial generators of cobordism groups of higher classifying spaces. We write $G_{(0)}$ or G_a to indicate some 0-form symmetry probed by 1-form a field. We write $G_{(1)}$ or G_b to indicate some 1-form symmetry probed by 2-form b field.

In this work, we compute the (co)bordism groups $\Omega_d^H(\mathrm{B}\mathbb{G})$ ($\mathrm{TP}_d(H\times\mathbb{G})$) for $H = \mathrm{O/SO/Spin/Pin}^{\pm}$ and several $\mathbb{G}$, we also consider $\Omega_d^{\mathbb{G}}$ where $\mathrm{B}\mathbb{G}$ is the total space of the nontrivial fibration with base space BO and fiber $\mathrm{B}^2\mathbb{Z}_2$ in section 4.1.

If there is a nontrivial group action between H and $\mathbb{G}$ (let us denote the action as the semi-direct product $\ltimes$), or if there is a shared common normal subgroup N_{shared} or sub-higher-group between H and $\mathbb{G}$, then we can generalize our above proposal eqn. (1.1) and eqn. (1.3) to

$$\left\{\begin{array}{c}\text{deformation classes of reflection positive}\\ \text{invertible } n\text{-dimensional extended topological}\\ \text{field theories with a symmetry group } (\frac{H_n\ltimes\mathbb{G}}{N_{\text{shared}}})\end{array}\right\}\cong[MT(\frac{H\ltimes\mathbb{G}}{N_{\text{shared}}}),\Sigma^{n+1}I\mathbb{Z}]_{\text{tors}}. \tag{1.4}$$

and

$$\mathrm{TP}_n(\frac{H\ltimes\mathbb{G}}{N_{\text{shared}}})\equiv[MT(\frac{H\ltimes\mathbb{G}}{N_{\text{shared}}}),\Sigma^{n+1}I\mathbb{Z}]. \tag{1.5}$$

For readers who wishes to explore other physics stories and some introduction materials, we suggest to look at the introduction of [25] and other shorter articles [18–20, 32, 33]. In particular, Ref. [33] provides a condensed matter interpretation of higher symmetries. We also encourage readers to read the Section I to III of [18].

We will explore the generic manifold with H-structure, including the orientable H = SO, Spin, etc., or unorientable H = O, $\mathrm{Pin}^{\pm}$. In physics, the quantum system that can be put on an unorientable H manifold implies that there is a time-reversal symmetry or a reflection symmetry (commonly termed the parity symmetry in an odd dimensional space). Physicists can find the introduction materials on the reflection symmetry and unorientable manifolds in Ref. [34, 35].

For readers who wishes to explore other mathematical introductory materials, we suggest to look at the [4, 36] and Appendices of [25].

Readers may be also interested in other recent work along the cobordism theory applications to physics [37] [38] [39] [40].

1.3 Mathematical preliminaries

In this subsection, we review the basics of bordism theory and possible generalizations.

Definition 1. If H is a group with a group homomorphism $\rho: H\to \mathrm{O}$, V is a vector bundle over M with a metric, then an H-structure on V is a principal H-bundle P over M, together with an isomorphism of bundles $P\times_H \mathrm{O}\xrightarrow{\sim}\mathcal{B}_{\mathrm{O}}(V)$ where $P\times_H\mathrm{O}$ is the quotient $(P\times\mathrm{O})/H$ where H acts freely on right of $P\times\mathrm{O}$ by

$$(p,g)\cdot h=(p\cdot h,\rho(h)^{-1}g),\quad p\in P,\quad g\in\mathrm{O},\quad h\in H$$

and $\mathcal{B}_{\mathrm{O}}(V)$ is the orthonormal frame bundle of V.

In particular, if $V = TM$, then an H-structure on TM is also called a tangential H-structure (or an H-structure) on M. Here we assume the H-structures are defined on the tangent bundles instead of normal bundles.

Below we consider manifolds with a metric.

Any manifold admits an O-structure, a manifold M admits an SO-structure if and only if $w_1(TM) = 0$, a manifold admits a Spin structure if and only if $w_1(TM) = w_2(TM) = 0$, a manifold admits a Pin^+ structure if and only if $w_2(TM) = 0$, a manifold admits a Pin^- structure if and only if $w_2(TM) + w_1(TM)^2 = 0$. Here $w_i(TM)$ is the i-th Stiefel-Whitney class of the tangent bundle of M.

Moreover, we can consider manifolds equipped with a map to a fixed topological space X, we are interested in the case when X is an Eilenberg-MacLane space since

$$[M, K(G,n)] = \mathrm{H}^n(M, G)$$

where the left hand side is the group of homotopy classes of maps from M to $K(G,n)$, the right hand side is the n-th cohomology group of M with coefficients in G.

Definition 2. Let H be a group, X be a fixed topological space, we can define an abelian group

$$\begin{aligned}\Omega_n^H(X) := \{(M,f)|&M \text{ is a closed}\\ &n\text{-manifold with } H\text{-structure, } f: M \to X \text{ is a map}\}/\text{bordism}.\end{aligned} \tag{1.6}$$

where bordism is an equivalence relation, namely, (M,f) and (M',f') are bordant if there exists a compact $n+1$-manifold N with H-structure and a map $h: N \to X$ such that the boundary of N is the disjoint union of M and M', the H-structures on M and M' are induced from the H-structure on N and $h|_M = f$, $h|_{M'} = f'$.

In particular, when $X = \mathrm{B}^2\mathbb{Z}_n$, $f: M \to \mathrm{B}^2\mathbb{Z}_n$ is a cohomology class in $\mathrm{H}^2(M, \mathbb{Z}_n)$. When $X = \mathrm{B}G$, with G is a Lie group or a finite group (viewed as a Lie group with discrete topology), then $f: M \to \mathrm{B}G$ is a principal G-bundle over M.

To explain our notation, here $\mathrm{B}G$ is a classifying space of G, and $\mathrm{B}^2\mathbb{Z}_n$ is a higher classifying space (Eilenberg-MacLane space $K(\mathbb{Z}_n, 2)$) of $\mathbb{Z}_n$.

In the particular case that $H = \mathrm{O}$ and X is a point, this definition 2 coincides with Thom's original definition.

In this article, we study the cases in which $H = \mathrm{O}/\mathrm{SO}/\mathrm{Spin}/\mathrm{Pin}^{\pm}$, and X is a higher classifying space, or more complicated cases.

If $\Omega_n^H(X) = G_1 \times G_2 \times \cdots \times G_r$ where G_i are cyclic groups, then group homomorphisms $\phi_i : \Omega_n^H(X) \to G_i$ form a complete set of *bordism invariants* if $\phi = (\phi_1, \phi_2, \dots, \phi_r) : \Omega_n^H(X) \to G_1 \times G_2 \times \cdots \times G_r$ is a group isomorphism.

Elements of $\Omega_n^H(X)$ are *manifold generators* if their images in $G_1 \times G_2 \times \cdots \times G_r$ under ϕ generate $G_1 \times G_2 \times \cdots \times G_r$.

We first introduce several concepts which are important for bordism theory:

Thom space: Let $V \to Y$ be a real vector bundle, and fix a Euclidean metric. The Thom space $\mathrm{Thom}(Y;V)$ is the quotient $D(V)/S(V)$ where $D(V)$ is the unit disk bundle and $S(V)$ is the unit

sphere bundle. Thom spaces satisfy

$$\begin{aligned}
\mathrm{Thom}(X \times Y; V \times W) &= \mathrm{Thom}(X; V) \wedge \mathrm{Thom}(Y; W), \\
\mathrm{Thom}(X, V \oplus \underline{\mathbb{R}^n}) &= \Sigma^n \mathrm{Thom}(X; V), \\
\mathrm{Thom}(X, \underline{\mathbb{R}^n}) &= \Sigma^n X_+
\end{aligned} \tag{1.7}$$

where $V \to X$ and $W \to Y$ are real vector bundles, $\underline{\mathbb{R}^n}$ is the trivial real vector bundle of dimension n, Σ is the suspension, X_+ is the disjoint union of X and a point.

We follow the definition of Thom spectrum and Madsen-Tillmann spectrum given in [2].

Thom spectrum [1]: MH is the Thom spectrum of the group H, it is the spectrification (see 2.2) of the prespectrum whose n-th space is $MH(n) = \mathrm{Thom}(\mathrm{B}H(n); V_n)$, and V_n is the induced vector bundle (of dimension n) by the map $\mathrm{B}H(n) \to \mathrm{BO}(n)$.

In other words, $MH = \mathrm{Thom}(\mathrm{B}H; V)$, where V is the induced virtual bundle (of dimension 0) by the map $\mathrm{B}H \to \mathrm{BO}$.

Madsen-Tillmann spectrum [3]: MTH is the Madsen-Tillmann spectrum of the group H, it is the colimit of $\Sigma^n MTH(n)$, where $MTH(n) = \mathrm{Thom}(\mathrm{B}H(n); -V_n)$, and V_n is the induced vector bundle (of dimension n) by the map $\mathrm{B}H(n) \to \mathrm{BO}(n)$. The virtual Thom spectrum $MTH(n)$ is the spectrification (see 2.2) of the prespectrum whose $(n+q)$-th space is $\mathrm{Thom}(\mathrm{B}H(n, n+q), Q_q)$ where $\mathrm{B}H(n, n+q)$ is the pullback

$$\begin{array}{ccc}
\mathrm{B}H(n, n+q) & \dashrightarrow & \mathrm{B}H(n) \\
\downarrow & & \downarrow \\
Gr_n(\mathbb{R}^{n+q}) & \longrightarrow & \mathrm{BO}(n)
\end{array} \tag{1.8}$$

and there is a direct sum $\underline{\mathbb{R}^{n+q}} = V_n \oplus Q_q$ of vector bundles over $Gr_n(\mathbb{R}^{n+q})$ and, by pullback, over $\mathrm{B}H(n, n+q)$ where $\underline{\mathbb{R}^{n+q}}$ is the trivial real vector bundle of dimension $n+q$.

In other words, $MTH = \mathrm{Thom}(\mathrm{B}H; -V)$, where V is the induced virtual bundle (of dimension 0) by the map $\mathrm{B}H \to \mathrm{BO}$.

Here Ω is the loop space, Σ is the suspension.

Note: "T" in MTH denotes that the H-structures are on tangent bundles instead of normal bundles.

(Co)bordism theory is a generalized (co)homology theory which is represented by a spectrum by the Brown representability theorem.

In fact, it is represented by Thom spectrum due to the Pontryagin-Thom isomorphism:

$$\begin{aligned}
\pi_n(MTH) &= \Omega_n^H \text{ the cobordism group of } n\text{-manifolds with tangential } H\text{-structure}, \\
\pi_n(MH) &= \Omega_n^{\nu H} \text{ the cobordism group of } n\text{-manifolds with normal } H\text{-structure}
\end{aligned} \tag{1.9}$$

In the case when tangential H-structure is the same as normal H'-structure, the relevant Thom spectra are weakly equivalent. In particular, $MT\mathrm{O} \simeq M\mathrm{O}$, $MT\mathrm{SO} \simeq M\mathrm{SO}$, $MT\mathrm{Spin} \simeq M\mathrm{Spin}$, $MT\mathrm{Pin}^+ \simeq M\mathrm{Pin}^-$, $MT\mathrm{Pin}^- \simeq M\mathrm{Pin}^+$.

$\mathrm{Pin}^\pm$ cobordism groups are not rings, though they are modules over the Spin cobordism ring.

By the generalized Pontryagin-Thom construction, for X a topological space, then the group of H-bordism classes of H-manifolds in X is isomorphic to the generalized homology of X with coefficients in MTH:

$$\Omega_d^H(X) = \pi_d(MTH \wedge X_+) = MTH_d(X) \tag{1.10}$$

where $\pi_d(MTH \wedge X_+)$ is the d-th stable homotopy group of the spectrum $MTH \wedge X_+$. The d-th stable homotopy group of a spectrum M is

$$\pi_d(M) = \mathrm{colim}_{k\to\infty}\pi_{d+k}M_k. \tag{1.11}$$

So the computation of the bordism group $\Omega_d^H(X)$ is the same as the computation of the stable homotopy group of the spectrum $MTH \wedge X_+$ which can be computed by Adams spectral sequence method.

Next, we introduce the Thom isomorphism [1]: Let $p : E \to B$ be a real vector bundle of rank n. Then there is an isomorphism, called Thom isomorphism

$$\Phi : \mathrm{H}^k(B, \mathbb{Z}_2) \to \tilde{\mathrm{H}}^{k+n}(T(E), \mathbb{Z}_2) \tag{1.12}$$

where $\tilde{\mathrm{H}}$ is the reduced cohomology, $T(E) = \mathrm{Thom}(E; B)$ is the Thom space and

$$\Phi(b) = p^*(b) \cup U \tag{1.13}$$

where U is the Thom class. We can define the i-th Stiefel-Whitney class of the vector bundle $p : E \to B$ by

$$w_i(p) = \Phi^{-1}(\mathrm{Sq}^i U) \tag{1.14}$$

where Sq is the Steenrod square.

1.4 Basics of Higher Symmetries and Higher Anomalies of Quantum Field Theory for Physicists and Mathematicians

In order to obtain a complete classification of 't Hooft anomalies of quantum field theories (QFTs), we aim to first identify the relevant (if not all of) global symmetry G (here we will abuse the notation to have G including the higher symmetry $\mathbb{G}$) of QFTs. Then we couple the QFTs to classical background-symmetric gauge field of G. Then we try to detect the possible obstructions of such coupling [13]. Such obstructions, known as the obstruction of gauging the global symmetry, are termed " 't Hooft anomalies" in QFT. In the literature, when people refer to "anomalies," however, they can means several related but different issues. To fix our terminology, we refer "anomalies" to be one of the followings:

1. Classical global symmetry is violated at the quantum theory, such that the classical global symmetry fails to survive as a quantum global symmetry, e.g. the original Adler-Bell-Jackiw (ABJ) anomaly [21, 22].

2. Quantum global symmetry is well-defined and preserved (for the Hamiltonian or path integral Lagrangian formulation of quantum theory). Namely, global symmetry is sensible, not only at a classical theory (if there is any classical description), but also for a quantum theory. However, there is an obstruction to gauge the global symmetry. Specifically, we can detect a certain obstruction to even *weakly gauge* the symmetry or couple the symmetry to a *non-dynamical background probed gauge field.* (We may abbreviate this background field as "bgd.field.") This is known as "'t Hooft anomaly," or sometimes regarded as a "weakly gauged anomaly" in condensed matter. Namely, the partition function $\mathbf{Z}$ does not sum over background gauge connections, but only fix a background gauge connection and only depend on the background gauge connection as a classical field (as a classical coupling constant). Say if the background gauge connection is A, the partition function is $\mathbf{Z}[A]$ depending on A. Normally, the $\mathbf{Z}[A]$ on a closed manifold in its own dimension is an invertible topological QFT (iTQFT), such that $\mathbf{Z}[A] = \exp(\mathrm{i}\theta(A))$ is a complex phase (thus physically meaningfully *invertible*) while its absolute value $|\mathbf{Z}[A]| = 1$ for any choice of background A.

3. Quantum global symmetry is well-defined and preserved (for the Hamiltonian or path integral Lagrangian formulation of quantum theory). However, once we promote the global symmetry to a gauge symmetry of the dynamical gauge theory, then the gauge theory becomes inconsistent. Some people call this as a "dynamical gauge anomaly" which makes a quantum theory inconsistent. Namely, the partition function $\mathbf{Z}$ after summing over dynamical gauge connections becomes inconsistent or ill-defined.

From now on, when we simply refer to "anomalies," we mean mostly "'t Hooft anomalies," which still have several intertwined interpretations:

Interpretation (1): In condensed matter physics, "'t Hooft anomalies" are known as the obstruction to *lattice-regularize* the global symmetry's quantum operator in a strictly local manner. By claiming local on a lattice or on a simplicial complex, we mean:
• on-site (e.g. on 0-simplex) for which 0-form symmetry operator acts on.
• on-link (e.g. on 1-simplex) for which 1-form symmetry operator acts on.
• on-plaquette (e.g. on 2-simplex) for which 2-form symmetry operator acts on.
...
• on n-simplex for which n-form symmetry operator acts on.
This obstruction is due to the symmetry-twists (See [Ref. [41–43]] for QFT-oriented discussion and references therein). This obstruction can be detected at high energy lattice scale (known as the ultraviolet [UV] in QFT). This "non-onsite symmetry" viewpoint is generically applicable to both, *perturbative* anomalies, and *non-perturbative global* anomalies:
• *Perturbative* anomalies — Characterized and captured by perturbative Feynman diagram calculations. Classified by an infinite integer $\mathbb{Z}$ class, known as the free (sub)group.
• *Non-perturbative or global* anomalies — Examples of global anomalies include the old and the new SU(2) anomalies [23, 24] (here we mean their 't Hooft anomaly analogs if we view the SU(2) gauge field as a non-dynamical classical background field) and the global gravitational anomalies [44]. These are classified by a finite group $\mathbb{Z}_n$ class, known as the torsion (sub)group.

These anomalies are sensitive to the underlying UV-completion not only of fermionic systems,

but also of bosonic systems [42, 45–47]. We term the anomalies of QFT whose UV-completion requires only the bosonic degrees of freedom as bosonic anomalies [45]. While we term those must require fermionic degrees of freedom as fermionic anomalies.

Interpretation (2): In QFTs, the obstruction is on the impossibility of adding any counter term in its own dimension (d-d) in order to absorb a one-higher-dimensional counter term (e.g. $(d+1)$d topological term) due to background G-field [48]. This is named the "anomaly-inflow [14]." The $(d+1)$d topological term is known as the $(d+1)$d SPTs in condensed matter physics [5, 49].

Interpretation (3): In math, the dd anomalies can be systematically captured by $(d+1)$d topological invariants [23] known as bordism invariants [4, 26, 27, 50].

• Bosonic anomalies or bosonic SPTs are normally characterized by topological terms detected via manifolds with $H = \mathrm{SO}$ (orientable) or O (unorientable) structures.
• Fermionic anomalies or fermionic SPTs are normally characterized by topological terms detected via manifolds with $H = \mathrm{Spin}$ (orientable) or $\mathrm{Pin}^{\pm}$ (unorientable) structures.

Below we summarize the higher symmetry $\mathbb{G}$ systematically introduced in [9].

(i) Higher symmetries and higher anomalies: The ordinary 0-form global symmetry has a *charged* object of 0d measured by the *charge* operator of $(d-1)$d. The generalized q-form global symmetry is introduced by Ref. [9]. A *charged* object of qd is measured by the *charge* operator of $(d-q-1)$d (i.e. codimension-$(q+1)$). This concept turns out to be powerful to detect new anomalies, e.g. the pure SU(N)-YM at $\theta = \pi$ has a mixed anomaly between 0-form time-reversal symmetry $\mathbb{Z}_2^T$ and 1-form center symmetry $\mathbb{Z}_{\mathrm{N},[1]}$ at an even integer N, firstly discovered in a remarkable work [Ref. [16]].

(ii) Relate (higher)-SPTs to (higher)-topological invariants: In the condensed matter literature, based on the earlier discussion on the *symmetry twist*, it has been recognized that the classical background-field partition function under the symmetry twist, called $\mathbf{Z}_{\text{sym.twist}}$ in $(d+1)$d can be regarded as the partition function of $(d+1)$d SPTs $\mathbf{Z}_{\text{SPTs}}$. These descriptions are applicable to both low-energy infrared (IR) field theory, but also to the UV-regulated SPTs on a lattice, see [Ref. [26, 41, 42]] and References therein. Schematically, we follow the framework of [42],

$$\mathbf{Z}^{(d+1)\text{d}}_{\text{sym.twist}} = \mathbf{Z}^{(d+1)\text{d}}_{\text{SPTs}} = \mathbf{Z}^{(d+1)\text{d}}_{\text{topo.inv}} = \mathbf{Z}^{(d+1)\text{d}}_{\text{Cobordism.inv}} \longleftrightarrow d\text{d-(higher) 't Hooft anomaly.} \tag{1.15}$$

In general, the partition function $\mathbf{Z}_{\text{sym.twist}} = \mathbf{Z}_{\text{SPTs}}[A_1, B_2, w_i, \dots]$ is a functional containing background gauge fields of 1-form A_1, 2-form B_2 or higher forms; and can contain characteristic classes [51] such as the i-th Stiefel-Whitney class (w_i) and other geometric probes such as gravitational background fields, e.g. a gravitational Chern-Simons 3-form $\mathrm{CS}_3(\Gamma)$ involving the Levi-Civita connection or the spin connection Γ. For our convention, we use the capital letters $(A, B, ...)$ to denote *non-dynamical background gauge* fields (which, however, later they may or may not be dynamically gauged), while the little letters $(a, b, ...)$ to denote *dynamical gauge* fields.
More generally,
• For the ordinary 0-form symmetry, we may couple the charged 0d point operator to 1-form background gauge field (so the symmetry-twist occurs in the Poincaré dual codimension-1 sub-spacetime [dd] of SPTs).

• For the 1-form symmetry, we may couple the charged 1d line operator to 2-form background gauge field (so the symmetry-twist occurs in the Poincaré dual codimension-2 sub-spacetime [$(d-1)$d] of SPTs).
• For the q-form symmetry, we may couple the *charged* qd extended operator to $(q+1)$-form background gauge field. The *charged* qd extended operator can be measured by another *charge* operator of codimension-$(q+1)$ [i.e. $(d-q)$d].
In summary, for the q-dimensional symmetry, we use the following terminology:

($\diamondsuit$ 1): *Charged* object: The *charged* q-dimensional extended operator as the q-dimensional-symmetry generator which is being measured by a symmetry generator.

($\diamondsuit$ 2): *Charge* operator: The corresponding *charge* operator of codimension-$(q+1)$ [i.e. $(d-q)$-dimension] which measures the q-dimensional-symmetry charged object.

So the symmetry-twist can be interpreted as the occurrence of the codimension-$(q+1)$ *charge* operator. In other words, the symmetry-twist happens at a Poincaré dual codimension-$(q+1)$ sub-spacetime [$(d-q)$d] of SPTs. We shall view the measurement of a *charged* qd extended object, happening at any q-dimensional intersection between the $(q+1)$d form background gauge field and the codimension-$(q+1)$ symmetry-twist or *charge* operator of this SPT vacua.

By higher-SPTs, we mean SPTs protected by higher symmetries (for generic q, especially for any SPTs with at least a symmetry of $q>0$). So our principle above is applicable to higher-SPTs [12,30]. In the following of this article, thanks to (1.15), we can interchange the usages and interpretations of "higher SPTs $\mathbf{Z}_{\text{SPTs}}$," "higher topological terms due to symmetry-twist $\mathbf{Z}^{(d+1)\text{d}}_{\text{sym.twist}}$," "higher topological invariants $\mathbf{Z}^{(d+1)\text{d}}_{\text{topo.inv}}$" or "bordism invariants $\mathbf{Z}^{(d+1)\text{d}}_{\text{Cobordism.inv}}$" in $(d+1)$d. They are all physically equivalent, and can uniquely determine a dd higher anomaly: if we study the anomaly of any boundary theory of the $(d+1)$d higher SPTs living on a manifold with dd boundary. Thus, we regard all of them as physically tightly-related given by (1.15). By turning on the classical background probed field (denoted as "bgd.field" in (1.16)) coupled to dd QFT, under the symmetry transformation (i.e. symmetry twist), its partition function $\mathbf{Z}^{d\text{d}}_{\text{QFT}}$ can be *shifted*

$$\mathbf{Z}^{d\text{d}}_{\text{QFT}}\Big|_{\text{bgd.field}=0} \longrightarrow \mathbf{Z}^{d\text{d}}_{\text{QFT}}\Big|_{\text{bgd.field}\neq 0} \cdot \mathbf{Z}^{(d+1)\text{d}}_{\text{SPTs}}(\text{bgd.field}), \tag{1.16}$$

to detect the underlying $(d+1)$d topological terms/counter term/SPTs, namely the $(d+1)$d partition function $\mathbf{Z}^{(d+1)\text{d}}_{\text{SPTs}}$. To check whether the underlying $(d+1)$d SPTs really specifies a true dd 't Hooft anomaly unremovable from dd *counter term*, it means that $\mathbf{Z}^{(d+1)\text{d}}_{\text{SPTs}}(\text{bgd.field})$ cannot be absorbed by a lower-dimensional SPTs $\mathbf{Z}^{d\text{d}}_{\text{SPTs}}(\text{bgd.field})$, namely

$$\mathbf{Z}^{d\text{d}}_{\text{QFT}}\Big|_{\text{bgd.field}} \cdot \mathbf{Z}^{(d+1)\text{d}}_{\text{SPTs}}(\text{bgd.field}) \neq \mathbf{Z}^{d\text{d}}_{\text{QFT}}\Big|_{\text{bgd.field}} \cdot \mathbf{Z}^{d\text{d}}_{\text{SPTs}}(\text{bgd.field}). \tag{1.17}$$

Readers can find related materials in [18,32].

1.5 The convention of notations

We explain the convention for our notations and terminology below. Most of our conventions follow [4] and [25].

- We denote O the stable orthogonal group, SO the stable special orthogonal group, Spin the stable spin group, and $\mathrm{Pin}^{\pm}$ the two ways of $\mathbb{Z}_2$ extension (related to the time reversal symmetry) of Spin group.

- $\mathbb{Z}_n$ is the finite cyclic group of order n, n is a positive integer.

- A map between topological spaces is always assumed to be continuous.

- For a (pointed) topological space X, Σ denotes a suspension $\Sigma X = S^1 \wedge X = (S^1 \times X)/(S^1 \vee X)$ where $\wedge$ and $\vee$ are smash product and wedge sum (one point union) of pointed topological spaces respectively. For a graded algebra A, $A = \bigoplus_i A_i$, ΣA is the graded algebra defined by $\Sigma A = \bigoplus_i (\Sigma A)_i$ where $(\Sigma A)_i = A_{i-1}$.

- For a (pointed) topological space X with the base point x_0, ΩX is the loop space of X:
$$\Omega X = \{\gamma : I \to X \text{ continuous} | \gamma(0) = \gamma(1) = x_0\}. \tag{1.18}$$

- Let R be a ring, M a topological space, $\mathrm{H}^*(M, R)$ is the cohomology ring of M with coefficients in R.

- We will abbreviate the cup product $x \cup y$ by xy.

- If M_d (or simply M) is a d-dimensional manifold, then TM_d (or simply TM) is the tangent bundle over M_d (or M).

- Rank r real (complex) vector bundle V is a bundle with fibers being real (complex) vector spaces of real (complex) dimension r.

- $w_i(V)$ is the i-th Stiefel-Whitney class of a real vector bundle V (which may be also complex rank r but considered as real rank $2r$).

- $p_i(V)$ is the i-th Pontryagin class of a real vector bundle V.

- $c_i(V)$ is the i-th Chern class of a complex vector bundle V. Pontryagin classes are closely related to Chern classes via complexification:
$$p_i(V) = (-1)^i c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C}) \tag{1.19}$$
where $V \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of the real vector bundle V. The relation between Pontryagin classes and Stiefel-Whitney classes is
$$p_i(V) = w_{2i}(V)^2 \mod 2. \tag{1.20}$$

- For a top degree cohomology class we often suppress explicit integration over the manifold (i.e. pairing with the fundamental class $[M]$). If M is orientable, then $[M]$ has coefficients in $\mathbb{Z}$. If M is non-orientable, then $[M]$ has coefficients in $\mathbb{Z}_2$.

- If x is an element of a graded vector space, $|x|$ denotes the degree of x.

- For an odd prime p and a non-negatively and integrally graded vector space V over $\mathbb{Z}_p$, let V^{even} and V^{odd} be even and odd graded parts of V . The free algebra $F_{\mathbb{Z}_p}[V]$ generated by the graded vector space V is the tensor product of the polynomial algebra on V^{even} and the exterior algebra on V^{odd}:
$$F_{\mathbb{Z}_p}[V] = \mathbb{Z}_p[V^{\text{even}}] \otimes \Lambda_{\mathbb{Z}_p}(V^{\text{odd}}). \tag{1.21}$$
We sometimes replace the vector space with a set of bases of it.

- $\mathcal{A}_p$ denotes the mod p Steenrod algebra where p is a prime.
- Sq^n is the n-th Steenrod square, it is an element of $\mathcal{A}_2$.
- $\mathcal{A}_2(1)$ denotes the subalgebra of $\mathcal{A}_2$ generated by Sq^1 and Sq^2.
- $\beta_{(n,m)} : \mathrm{H}^*(-,\mathbb{Z}_m) \to \mathrm{H}^{*+1}(-,\mathbb{Z}_n)$ is the Bockstein homomorphism associated to the extension $\mathbb{Z}_n \overset{\cdot m}{\to} \mathbb{Z}_{nm} \to \mathbb{Z}_m$, when $n = m = p$ is a prime, it is an element of $\mathcal{A}_p$. If $p = 2$, then $\beta_{(2,2)} = \mathrm{Sq}^1$.
- $P_p^n : \mathrm{H}^*(-,\mathbb{Z}_p) \to \mathrm{H}^{*+2n(p-1)}(-,\mathbb{Z}_p)$ is the n-th Steenrod power, it is an element of $\mathcal{A}_p$ where p is an odd prime. For odd primes p, we only consider $p = 3$, so we abbreviate P_3^n by P^n.
- $\mathcal{P}_2$ is the Pontryagin square operation $\mathrm{H}^{2i}(M,\mathbb{Z}_{2^k}) \to \mathrm{H}^{4i}(M,\mathbb{Z}_{2^{k+1}})$. Explicitly, $\mathcal{P}_2$ is given by

$$\mathcal{P}_2(x) = x \cup x + x \underset{1}{\cup} \delta x \mod 2^{k+1} \tag{1.22}$$

and it satisfies

$$\mathcal{P}_2(x) = x \cup x \mod 2^k. \tag{1.23}$$

Here $\underset{1}{\cup}$ is the higher cup product.

- Postnikov square $\mathfrak{P}_3 : \mathrm{H}^2(-,\mathbb{Z}_{3^k}) \to \mathrm{H}^5(-,\mathbb{Z}_{3^{k+1}})$ is given by

$$\mathfrak{P}_3(u) = \beta_{(3^{k+1},3^k)}(u \cup u) \tag{1.24}$$

where $\beta_{(3^{k+1},3^k)}$ is the Bockstein homomorphism associated to $0 \to \mathbb{Z}_{3^{k+1}} \to \mathbb{Z}_{3^{2k+1}} \to \mathbb{Z}_{3^k} \to 0$.

- For a finitely generated abelian group G and a prime p, $G_p^{\wedge} = \lim_n G/p^n G$ is the p-completion of G.
- For a topological space M, $\pi_d(M)$ is the d-th (ordinary) homotopy group of M.
- For an abelian group G, the Eilenberg-MacLane space $K(G,n)$ is a space with homotopy groups satisfying

$$\pi_i K(G,n) = \begin{cases} G, & i = n. \\ 0, & i \neq n. \end{cases} \tag{1.25}$$

- Let X, Y be topological spaces, $[X,Y]$ is the set of homotopy classes of maps from X to Y.
- Let G be a group, the classifying space of G, $\mathrm{B}G$ is a topological space such that

$$[X,\mathrm{B}G] = \{\text{isomorphism classes of principal } G\text{-bundles over } X\} \tag{1.26}$$

for any topological space X. In particular, if G is an abelian group, then $\mathrm{B}G$ is a group.

- There is a vector bundle associated to a principal G-bundle P_G: $P_G \times_G V = (P_G \times V)/G$ which is the quotient of $P_G \times V$ by the right G-action

$$(p,v)g = (pg, g^{-1}v) \tag{1.27}$$

where V is the vector space which G acts on. For characteristic classes of a principal G-bundle, we mean the characteristic classes of the associated vector bundle.

1.6 Tables and Summary of Some Co/Bordism Groups

Below we use the following notations, all cohomology class are pulled back to the d-manifold M along the maps given in the definition of cobordism groups:
• w_i is the Stiefel-Whitney class of the tangent bundle of M,
• a is the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$,
• a' is the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_3, \mathbb{Z}_3)$, $b' = \beta_{(3,3)}a'$,
• x_2 is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \mathrm{Sq}^1 x_2$, $x_5 = \mathrm{Sq}^2 x_3$,
• x_2' is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3)$, $x_3' = \beta_{(3,3)}x_2'$,
• $w_i' = w_i(\mathrm{PSU}(2)) \in \mathrm{H}^i(\mathrm{BPSU}(2), \mathbb{Z}_2)$ is the Stiefel-Whitney class of the principal PSU(2) bundle,
• $c_i = c_i(\mathrm{PSU}(3)) \in \mathrm{H}^{2i}(\mathrm{BPSU}(3), \mathbb{Z})$ is the Chern class of the principal PSU(3) bundle,
• $z_2 = w_2(\mathrm{PSU}(3)) \in \mathrm{H}^2(\mathrm{BPSU}(3), \mathbb{Z}_3)$ is the generalized Stiefel-Whitney class of the principal PSU(3) bundle, $z_3 = \beta_{(3,3)}z_2$.
• $\mathcal{P}_2$ is the Pontryagin square (see 1.5).
• $\mathfrak{P}_3$ is the Postnikov square (see 1.5).
Conventions: All product between cohomology classes are cup product, product between a cohomology class x and $\tilde\eta$ (or Arf, ABK, etc) means the value of $\tilde\eta$ (or Arf, ABK, etc) on the submanifold of M which represents the Poincaré dual of x.

$\Omega_d^H(-)$	$\mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}^2\mathbb{Z}_3$	BPSU(2)	BPSU(3)	$\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3$	$\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3$
2 SO	$\mathbb{Z}_2$: x_2	$\mathbb{Z}_3$: x_2'	$\mathbb{Z}_2$: w_2'	$\mathbb{Z}_3$: z_2	$\mathbb{Z}_2$: x_2	$\mathbb{Z}_3$: x_2'	$\mathbb{Z}_2^2$: w_2', x_2	$\mathbb{Z}_3^2$: x_2', z_2
2 Spin	$\mathbb{Z}_2^2$: x_2, Arf[3]	$\mathbb{Z}_2 \times \mathbb{Z}_3$: Arf, x_2'	$\mathbb{Z}_2^2$: w_2', Arf	$\mathbb{Z}_2 \times \mathbb{Z}_3$: Arf, z_2	$\mathbb{Z}_2^3$: x_2, Arf, $a\tilde\eta$[4]	$\mathbb{Z}_2 \times \mathbb{Z}_3$: Arf, x_2'	$\mathbb{Z}_2^3$: w_2', x_2, Arf	$\mathbb{Z}_2 \times \mathbb{Z}_3^2$: Arf, x_2', z_2
2 O	$\mathbb{Z}_2^2$: x_2, w_1^2	$\mathbb{Z}_2$: w_1^2	$\mathbb{Z}_2^2$: w_2', w_1^2	$\mathbb{Z}_2$: w_1^2	$\mathbb{Z}_2^3$: a^2, x_2, w_1^2	$\mathbb{Z}_2$: w_1^2	$\mathbb{Z}_2^3$: w_2', x_2, w_1^2	$\mathbb{Z}_2$: w_1^2
2 Pin$^+$	$\mathbb{Z}_2^2$: $x_2, w_1\tilde\eta$	$\mathbb{Z}_2$: $w_1\tilde\eta$	$\mathbb{Z}_2^2$: $w_2', w_1\tilde\eta$	$\mathbb{Z}_2$: $w_1\tilde\eta$	$\mathbb{Z}_2^3$: $w_1 a = a^2, x_2, w_1\tilde\eta$	$\mathbb{Z}_2$: $w_1\tilde\eta$	$\mathbb{Z}_2^3$: $w_2', x_2, w_1\tilde\eta$	$\mathbb{Z}_2$: $w_1\tilde\eta$
2 Pin$^-$	$\mathbb{Z}_2 \times \mathbb{Z}_8$: x_2, ABK[5]	$\mathbb{Z}_8$: ABK	$\mathbb{Z}_2 \times \mathbb{Z}_8$: w_2', ABK	$\mathbb{Z}_8$: ABK	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$: $x_2, q(a)$[6], ABK	$\mathbb{Z}_8$: ABK	$\mathbb{Z}_2^2 \times \mathbb{Z}_8$: w_2', x_2, ABK	$\mathbb{Z}_8$: ABK

Table 1: $2d$ bordism groups.

[3] Arf is the Arf invariant of Spin 2-manifolds.
[4] $\tilde\eta$ is the "mod 2 index" of the 1d Dirac operator (#zero eigenvalues mod 2, no contribution from spectral asymmetry).
[5] ABK is the Arf-Brown-Kervaire invariant of Pin$^-$ 2-manifolds.
[6] Any 2-manifold Σ always admits a Pin$^-$ structure. Pin$^-$ structures are in one-to-one correspondence with quadratic enhancement

$$q : \mathrm{H}^1(\Sigma, \mathbb{Z}_2) \to \mathbb{Z}_4 \tag{1.28}$$

such that

$$q(x+y) - q(x) - q(y) = 2\int_\Sigma x \cup y \mod 4. \tag{1.29}$$

In particular:

$$q(x) = \int_\Sigma x \cup x \mod 2. \tag{1.30}$$

$\Omega_d^H(-)$	$\mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}^2\mathbb{Z}_3$	$\mathrm{BPSU}(2)$	$\mathrm{BPSU}(3)$	$\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3$	$\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3$
3 SO	0	0	0	0	$\mathbb{Z}_2^2$: ax_2, a^3	$\mathbb{Z}_3^2$: $a'b', a'x_2'$	0	0
3 Spin	0	0	0	0	$\mathbb{Z}_2\times\mathbb{Z}_8$: $ax_2, a\mathrm{ABK}$	$\mathbb{Z}_3^2$: $a'b', a'x_2'$	0	0
3 O	$\mathbb{Z}_2$: $x_3 = w_1x_2$	0	$\mathbb{Z}_2$: $w_3' = w_1w_2'$	0	$\mathbb{Z}_2^4$: $x_3 = w_1x_2, ax_2, aw_1^2, a^3$	0	$\mathbb{Z}_2^2$: $x_3 = w_1x_2, w_3' = w_1w_2'$	0
3 Pin^+	$\mathbb{Z}_2^2$: $w_1x_2 = x_3$, $w_1\mathrm{Arf}$	$\mathbb{Z}_2$: $w_1\mathrm{Arf}$	$\mathbb{Z}_2^2$: $w_1w_2' = w_3'$, $w_1\mathrm{Arf}$	$\mathbb{Z}_2$: $w_1\mathrm{Arf}$	$\mathbb{Z}_2^5$: $a^3, w_1x_2 = x_3, ax_2, w_1a\tilde{\eta}, w_1\mathrm{Arf}$	$\mathbb{Z}_2$: $w_1\mathrm{Arf}$	$\mathbb{Z}_2^3$: $w_1w_2' = w_3', w_1x_2 = x_3, w_1\mathrm{Arf}$	$\mathbb{Z}_2$: $w_1\mathrm{Arf}$
3 Pin^-	$\mathbb{Z}_2$: $w_1x_2 = x_3$	0	$\mathbb{Z}_2$: $w_1w_2' = w_3'$	0	$\mathbb{Z}_2^4$: $a^3, w_1^2a, x_3 = w_1x_2, ax_2$	0	$\mathbb{Z}_2^2$: $w_1w_2' = w_3'$, $w_1x_2 = x_3$	0

Table 2: $3d$ bordism groups.

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[7] $q_s : \mathrm{H}^2(M,\mathbb{Z}_2) \to \mathbb{Z}_4$ is a $\mathbb{Z}_4$ valued quadratic refinement (dependent on the choice of Pin^+ structure $s \in \mathrm{Pin}^+(M)$) of the intersection form

$$\langle , \rangle : \mathrm{H}^2(M,\mathbb{Z}_2)\times \mathrm{H}^2(M,\mathbb{Z}_2) \to \mathbb{Z}_2$$

i.e. so that $q_s(x+y) - q_s(x) - q_s(y) = 2\langle x, y\rangle \in \mathbb{Z}_4$ (in particular $q_s(x) = \langle x, x\rangle \mod 2$)

The space of Pin^+ structures is acted upon freely and transitively by $\mathrm{H}^1(M,\mathbb{Z}_2)$, and the dependence of q_s on the Pin^+ structure should satisfy

$$q_{s+h}(x) - q_s(x) = 2w_1(TM)hx, \text{ for any } h \in \mathrm{H}^1(M,\mathbb{Z}_2)$$

(note that any two quadratic functions differ by a linear function)

If $w_1(TM) = 0$, then $q_s(x)$ is independent on the Pin^+ structure $s \in \mathrm{Pin}^+(M)$, it reduces to $\mathcal{P}_2(x)$ where $\mathcal{P}_2(x)$ is the Pontryagin square of x.

[8] Here η is the usual Atiyah-Patodi-Singer eta-invariant of the 4d Dirac operator (="#zero eigenvalues + spectral asymmetry").

[9] one can also define this $\mathbb{Z}_4$ invariant as

$$(\eta_{\mathrm{SO}(3)} - 3\eta)/4 \in \mathbb{Z}_4 \quad (*)$$

where $\eta \in \mathbb{Z}_{16}$ is the (properly normalized) eta-invariant of the ordinary Dirac operator, and $\eta_{\mathrm{SO}(3)} \in \mathbb{Z}_{16}$ is the eta invariant of the twisted Dirac operator acting on the $S\otimes V_3$ where S is the spinor bundle and V_3 is the bundle associated to 3-dim representation of SO(3). Note that $(*)$ is well defined because $\eta_{\mathrm{SO}(3)} = 3\eta \mod 4$.

Note that on non-orientable manifold, if $w_2(V_3) = 0$, then since $w_1(V_3) = 0$, we also have $w_3(V_3) = 0$, hence V_3 is stably trivial, $\eta_{\mathrm{SO}(3)} = 3\eta$.

Also note that on oriented manifold one can use Atiyah-Patodi-Singer index theorem to show that (here the normalization of eta-invariants is such that η is an integer mod 16 on a general non-oriented 4-manifold)

$$\eta = -\frac{\sigma(M)}{2},$$

$$\eta_{\mathrm{SO}(3)} = -\frac{3\sigma(M)}{2} + 4p_1(\mathrm{SO}(3)).$$

So

$$(\eta_{\mathrm{SO}(3)} - 3\eta)/4 = p_1(\mathrm{SO}(3)) \mod 4 = \mathcal{P}_2(w_2(\mathrm{SO}(3))).$$

$q_s(w_2(\mathrm{SO}(3)))$ also reduces to $\mathcal{P}_2(w_2(\mathrm{SO}(3)))$ in the oriented case.

$\Omega_d^H(-)$	$\mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}^2\mathbb{Z}_3$	$\mathrm{BPSU}(2)$	$\mathrm{BPSU}(3)$	$\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3$	$\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3$
4 SO	$\mathbb{Z}\times\mathbb{Z}_4$: $\sigma, \mathcal{P}_2(x_2)$	$\mathbb{Z}\times\mathbb{Z}_3$: $\sigma, x_2'^2$	$\mathbb{Z}^2$: σ, p_1'	$\mathbb{Z}^2$: σ, c_2	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4$: σ, $ax_3 = a^2x_2$, $\mathcal{P}_2(x_2)$	$\mathbb{Z}\times\mathbb{Z}_3^2$: σ, $a'x_3' = b'x_2'$, $x_2'^2$	$\mathbb{Z}^2\times\mathbb{Z}_2\times\mathbb{Z}_4$: σ, p_1', $w_2'x_2, \mathcal{P}_2(x_2)$	$\mathbb{Z}^2\times\mathbb{Z}_3^2$: σ, c_2, $x_2'^2, x_2'z_2$
4 Spin	$\mathbb{Z}\times\mathbb{Z}_2$: $\frac{\sigma}{16}$, $\frac{\mathcal{P}_2(x_2)}{2}$	$\mathbb{Z}\times\mathbb{Z}_3$: $\frac{\sigma}{16}, x_2'^2$	$\mathbb{Z}^2$: $\frac{\sigma}{16}, \frac{p_1'}{2}$	$\mathbb{Z}^2$: $\frac{\sigma}{16}, c_2$	$\mathbb{Z}\times\mathbb{Z}_2^2$: $\frac{\sigma}{16}$, $ax_3 = a^2x_2$, $\frac{\mathcal{P}_2(x_2)}{2}$	$\mathbb{Z}\times\mathbb{Z}_3^2$: $\frac{\sigma}{16}$, $a'x_3' = b'x_2'$, $x_2'^2$	$\mathbb{Z}^2\times\mathbb{Z}_2^2$: $\frac{\sigma}{16}, \frac{p_1'}{2}$, $w_2'x_2$, $\frac{\mathcal{P}_2(x_2)}{2}$	$\mathbb{Z}^2\times\mathbb{Z}_3^2$: $\frac{\sigma}{16}, c_2$, $x_2'^2, x_2'z_2$
4 O	$\mathbb{Z}_2^4$: x_2^2, w_1^4, $w_1^2x_2, w_2^2$	$\mathbb{Z}_2^2$: w_1^4, w_2^2	$\mathbb{Z}_2^4$: $w_2'^2, w_1^4$, $w_1^2w_2', w_2^2$	$\mathbb{Z}_2^3$: w_1^4, w_2^2, $c_2(\mathrm{mod}\ 2)$	$\mathbb{Z}_2^8$: w_1^4, w_2^2, a^4, a^2x_2, ax_3, x_2^2, $w_1^2a^2, w_1^2x_2$	$\mathbb{Z}_2^2$: w_1^4, w_2^2	$\mathbb{Z}_2^7$: w_1^4, w_2^2, $x_2^2, w_2'^2$, $x_2w_1^2, w_2'w_1^2$, $w_2'x_2$	$\mathbb{Z}_2^3$: w_1^4, w_2^2, $c_2(\mathrm{mod}\ 2)$
4 Pin$^+$	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$: $q_s(x_2)$[7], η[8]	$\mathbb{Z}_{16}$: η	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$: $q_s(w_2')$[9], η	$\mathbb{Z}_2\times\mathbb{Z}_{16}$: $c_2(\mathrm{mod}\ 2)$, η	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8\times\mathbb{Z}_{16}$: $ax_3, w_1ax_2 = a^2x_2 + ax_3$, $q_s(x_2)$, $w_1a\mathrm{ABK}$, η	$\mathbb{Z}_{16}$: η	$\mathbb{Z}_4^2\times\mathbb{Z}_{16}\times \mathbb{Z}_2$: $q_s(w_2')$, $q_s(x_2)$, $\eta, w_2'x_2$	$\mathbb{Z}_2\times\mathbb{Z}_{16}$: $c_2(\mathrm{mod}\ 2)$, η
4 Pin$^-$	$\mathbb{Z}_2$: $w_1^2x_2$	0	$\mathbb{Z}_2$: $w_1^2w_2'$	$\mathbb{Z}_2$: $c_2(\mathrm{mod}\ 2)$	$\mathbb{Z}_2^3$: $w_1^2x_2, ax_3$, $w_1ax_2 = a^2x_2 + ax_3$	0	$\mathbb{Z}_2^3$: $w_1^2w_2'$, $w_1^2x_2, w_2'x_2$	$\mathbb{Z}_2$: $c_2(\mathrm{mod}\ 2)$

Table 3: $4d$ bordism groups.

In Section 4.1, we compute the topological terms (involving the cohomology classes of $\mathrm{B}^2\mathbb{Z}_2$) of $\Omega_5^{\mathbb{G}}$ where $\mathbb{G}$ is a 2-group with $G_a = \mathrm{O}$, $G_b = \mathbb{Z}_2$ We find that the term x_2w_3 (or x_3w_2) survives only for $\beta = 0, w_1^3$ (the Postnikov class $\beta \in \mathrm{H}^3(\mathrm{BO}, \mathbb{Z}_2) = \mathbb{Z}_2^3$ which is generated by w_1^3, w_1w_2, w_3). This term also appears in eq. 2.57 of [17].

[10] $\mathrm{CS}_3(TM) \equiv \mathrm{CS}_3^{(TM)}$ is the Chern-Simons 3-form of the tangent bundle.

[11] $\mathrm{CS}_3(\mathrm{SO}(3)) \equiv \mathrm{CS}_3^{(\mathrm{SO}(3))}$ is the Chern-Simons 3-form of the SO(3) gauge bundle.

[12] $\mathrm{CS}_3(\mathrm{PSU}(3)) \equiv \mathrm{CS}_3^{(\mathrm{PSU}(3))}$ is the Chern-Simons 3-form of the PSU(3) gauge bundle.

[13] $\mathrm{CS}_5(\mathrm{PSU}(3)) \equiv \mathrm{CS}_5^{(\mathrm{PSU}(3))}$ is the Chern-Simons 5-form of the PSU(3) gauge bundle.

$\Omega_d^H(-)$	$\mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}^2\mathbb{Z}_3$	BPSU(2)	BPSU(3)	$\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3$	$\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3$
5 SO	$\mathbb{Z}_2^2$: $x_5 = x_2x_3$, w_2w_3	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^2$: w_2w_3, $w_2'w_3'$	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^6$: ax_2^2, a^5, $x_5 = x_2x_3, a^3x_2$, w_2w_3, aw_2^2	$\mathbb{Z}_2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_9$: w_2w_3, $a'b'x_2'$, $a'x_2'^2$, $\mathfrak{P}_3(b')$	$\mathbb{Z}_2^4$: $w_2'w_3', x_5 = x_2x_3$, $w_3'x_2 = w_2'x_3$, w_2w_3	$\mathbb{Z}_2 \times \mathbb{Z}_3$: w_2w_3, $z_2x_3' = -z_3x_2'$
5 Spin	0	0	0	0	$\mathbb{Z}_2$: a^3x_2	$\mathbb{Z}_3^2 \times \mathbb{Z}_9$: $a'b'x_2'$, $a'x_2'^2$, $\mathfrak{P}_3(b')$	$\mathbb{Z}_2$: $w_3'x_2 = w_2'x_3$	$\mathbb{Z}_3$: $z_2x_3' = -z_3x_2'$
5 O	$\mathbb{Z}_2^4$: x_2x_3, x_5, $w_1^2x_3$, w_2w_3	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^3$: w_2w_3, $w_1^2w_3'$, $w_2'w_3'$	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^{12}$: a^5, a^2x_3, $a^3x_2, a^3w_1^2$, ax_2^2, aw_1^4, $ax_2w_1^2, aw_2^2$, $x_2x_3, w_1^2x_3$, x_5, w_2w_3	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^8$: $w_2'w_3', x_2w_3'$, $w_1^2w_3', w_2'x_3$, $x_2x_3, w_1^2x_3$, x_5, w_2w_3	$\mathbb{Z}_2$: w_2w_3
5 Pin^+	$\mathbb{Z}_2^2$: x_2x_3, $w_1^2x_3 = x_5$	0	$\mathbb{Z}_2$: $w_1^2w_3' = w_2'w_3'$	0	$\mathbb{Z}_2^7$: w_1^4a, $a^5 = w_1^2a^3$, $w_1^2x_3 = x_5$, x_2x_3, $w_1^2ax_2 = ax_2^2 + a^2x_3$, $w_1ax_3 = a^2x_3$, a^3x_2	0	$\mathbb{Z}_2^5$: $w_1^2w_3' = w_2'w_3'$, $w_1^2x_3 = x_5$, $x_2x_3, w_3'x_2$, $w_1w_2'x_2 = w_2'x_3 + w_3'x_2$	0
5 Pin^-	$\mathbb{Z}_2$: x_2x_3	0	0	0	$\mathbb{Z}_2^5$: $w_1^2a^3, x_2x_3$, $w_1^2ax_2$, $w_1ax_3 = a^2x_3$, a^3x_2	0	$\mathbb{Z}_2^3$: $x_2x_3, w_3'x_2$, $w_1w_2'x_2 = w_2'x_3 + w_3'x_2$	0

Table 4: $5d$ bordism groups.

$\mathrm{TP}_d(H\times -)$	$\mathrm{B}\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_3$	PSU(2)	PSU(3)	$\mathbb{Z}_2\times\mathrm{B}\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathrm{B}\mathbb{Z}_3$	$\mathrm{PSU}(2)\times\mathrm{B}\mathbb{Z}_2$	$\mathrm{PSU}(3)\times\mathrm{B}\mathbb{Z}_3$
2 SO	$\mathbb{Z}_2: x_2$	$\mathbb{Z}_3: x_2'$	$\mathbb{Z}_2: w_2'$	$\mathbb{Z}_3: z_2$	$\mathbb{Z}_2: x_2$	$\mathbb{Z}_3: x_2'$	$\mathbb{Z}_2^2: w_2', x_2$	$\mathbb{Z}_3^2: x_2', z_2$
2 Spin	$\mathbb{Z}_2^2: x_2, \mathrm{Arf}$	$\mathbb{Z}_2\times\mathbb{Z}_3: \mathrm{Arf}, x_2'$	$\mathbb{Z}_2^2: w_2', \mathrm{Arf}$	$\mathbb{Z}_2\times\mathbb{Z}_3: \mathrm{Arf}, z_2$	$\mathbb{Z}_2^3: x_2, \mathrm{Arf}, a\tilde\eta$	$\mathbb{Z}_2\times\mathbb{Z}_3: \mathrm{Arf}, x_2'$	$\mathbb{Z}_2^3: w_2', x_2, \mathrm{Arf}$	$\mathbb{Z}_2\times\mathbb{Z}_3^2: \mathrm{Arf}, x_2', z_2$
2 O	$\mathbb{Z}_2^2: x_2, w_1^2$	$\mathbb{Z}_2: w_1^2$	$\mathbb{Z}_2^2: w_2', w_1^2$	$\mathbb{Z}_2: w_1^2$	$\mathbb{Z}_2^3: a^2, x_2, w_1^2$	$\mathbb{Z}_2: w_1^2$	$\mathbb{Z}_2^3: w_2', x_2, w_1^2$	$\mathbb{Z}_2: w_1^2$
2 Pin^+	$\mathbb{Z}_2^2: x_2, w_1\tilde\eta$	$\mathbb{Z}_2: w_1\tilde\eta$	$\mathbb{Z}_2^2: w_2', w_1\tilde\eta$	$\mathbb{Z}_2: w_1\tilde\eta$	$\mathbb{Z}_2^3: w_1 a = a^2, x_2, w_1\tilde\eta$	$\mathbb{Z}_2: w_1\tilde\eta$	$\mathbb{Z}_2^3: w_2', x_2, w_1\tilde\eta$	$\mathbb{Z}_2: w_1\tilde\eta$
2 Pin^-	$\mathbb{Z}_2\times\mathbb{Z}_8: x_2, \mathrm{ABK}$	$\mathbb{Z}_8: \mathrm{ABK}$	$\mathbb{Z}_2\times\mathbb{Z}_8: w_2', \mathrm{ABK}$	$\mathbb{Z}_8: \mathrm{ABK}$	$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_8: x_2, q(a), \mathrm{ABK}$	$\mathbb{Z}_8: \mathrm{ABK}$	$\mathbb{Z}_2^2\times\mathbb{Z}_8: w_2', x_2, \mathrm{ABK}$	$\mathbb{Z}_8: \mathrm{ABK}$

Table 5: TP_2.

$\mathrm{TP}_d(H\times -)$	$\mathrm{B}\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_3$	PSU(2)	PSU(3)	$\mathbb{Z}_2\times\mathrm{B}\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathrm{B}\mathbb{Z}_3$	$\mathrm{PSU}(2)\times\mathrm{B}\mathbb{Z}_2$	$\mathrm{PSU}(3)\times\mathrm{B}\mathbb{Z}_3$
3 SO	$\mathbb{Z}: \frac{1}{3}\mathrm{CS}_3^{(TM)}$ [10]	$\mathbb{Z}: \frac{1}{3}\mathrm{CS}_3^{(TM)}$	$\mathbb{Z}^2: \frac{1}{3}\mathrm{CS}_3^{(TM)}, \mathrm{CS}_3^{(\mathrm{SO}(3))}$ [11]	$\mathbb{Z}^2: \frac{1}{3}\mathrm{CS}_3^{(TM)}, \mathrm{CS}_3^{(\mathrm{PSU}(3))}$ [12]	$\mathbb{Z}\times\mathbb{Z}_2^2: \frac{1}{3}\mathrm{CS}_3^{(TM)}, ax_2, a^3$	$\mathbb{Z}\times\mathbb{Z}_3^2: \frac{1}{3}\mathrm{CS}_3^{(TM)}, a'b', a'x_2'$	$\mathbb{Z}^2: \frac{1}{3}\mathrm{CS}_3^{(TM)}, \mathrm{CS}_3^{(\mathrm{SO}(3))}$	$\mathbb{Z}^2: \frac{1}{3}\mathrm{CS}_3^{(TM)}, \mathrm{CS}_3^{(\mathrm{PSU}(3))}$
3 Spin	$\mathbb{Z}: \frac{1}{48}\mathrm{CS}_3^{(TM)}$	$\mathbb{Z}: \frac{1}{48}\mathrm{CS}_3^{(TM)}$	$\mathbb{Z}^2: \frac{1}{48}\mathrm{CS}_3^{(TM)}, \frac{1}{2}\mathrm{CS}_3^{(\mathrm{SO}(3))}$	$\mathbb{Z}^2: \frac{1}{48}\mathrm{CS}_3^{(TM)}, \mathrm{CS}_3^{(\mathrm{PSU}(3))}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_8: \frac{1}{48}\mathrm{CS}_3^{(TM)}, ax_2, a\mathrm{ABK}$	$\mathbb{Z}\times\mathbb{Z}_3^2: \frac{1}{48}\mathrm{CS}_3^{(TM)}, a'b', a'x_2'$	$\mathbb{Z}^2: \frac{1}{48}\mathrm{CS}_3^{(TM)}, \frac{1}{2}\mathrm{CS}_3^{(\mathrm{SO}(3))}$	$\mathbb{Z}^2: \frac{1}{48}\mathrm{CS}_3^{(TM)}, \mathrm{CS}_3^{(\mathrm{PSU}(3))}$
3 O	$\mathbb{Z}_2: x_3 = w_1x_2$	0	$\mathbb{Z}_2: w_3' = w_1w_2'$	0	$\mathbb{Z}_2^4: x_3 = w_1x_2, ax_2, aw_1^2, a^3$	0	$\mathbb{Z}_2^2: x_3 = w_1x_2, w_3' = w_1w_2'$	0
3 Pin^+	$\mathbb{Z}_2^2: w_1x_2 = x_3, w_1\mathrm{Arf}$	$\mathbb{Z}_2: w_1\mathrm{Arf}$	$\mathbb{Z}_2^2: w_1w_2' = w_3', w_1\mathrm{Arf}$	$\mathbb{Z}_2: w_1\mathrm{Arf}$	$\mathbb{Z}_2^5: a^3, w_1x_2 = x_3, ax_2, w_1a\tilde\eta, w_1\mathrm{Arf}$	$\mathbb{Z}_2: w_1\mathrm{Arf}$	$\mathbb{Z}_2^3: w_1w_2' = w_3', w_1x_2 = x_3, w_1\mathrm{Arf}$	$\mathbb{Z}_2: w_1\mathrm{Arf}$
3 Pin^-	$\mathbb{Z}_2: w_1x_2 = x_3$	0	$\mathbb{Z}_2: w_1w_2' = w_3'$	0	$\mathbb{Z}_2^4: a^3, w_1^2a, x_3 = w_1x_2, ax_2$	0	$\mathbb{Z}_2^2: w_1w_2' = w_3', w_1x_2 = x_3$	0

Table 6: TP_3.

$\mathrm{TP}_d(H \times -)$	$\mathrm{B}\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_3$	PSU(2)	PSU(3)	$\mathbb{Z}_2 \times \mathrm{B}\mathbb{Z}_2$	$\mathbb{Z}_3 \times \mathrm{B}\mathbb{Z}_3$	PSU(2) $\times$ $\mathrm{B}\mathbb{Z}_2$	PSU(3) $\times$ $\mathrm{B}\mathbb{Z}_3$
4 SO	$\mathbb{Z}_4$: $\mathcal{P}_2(x_2)$	$\mathbb{Z}_3$: $x_2'^2$	0	0	$\mathbb{Z}_2 \times \mathbb{Z}_4$: $ax_3 = a^2x_2$, $\mathcal{P}_2(x_2)$	$\mathbb{Z}_3^2$: $a'x_3' = b'x_2'$, $x_2'^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$: $w_2'x_2, \mathcal{P}_2(x_2)$	$\mathbb{Z}_3^2$: $x_2'^2, x_2'z_2$
4 Spin	$\mathbb{Z}_2$: $\frac{\mathcal{P}_2(x_2)}{2}$	$\mathbb{Z}_3$: $x_2'^2$	0	0	$\mathbb{Z}_2^2$: $ax_3 = a^2x_2$, $\frac{\mathcal{P}_2(x_2)}{2}$	$\mathbb{Z}_3^2$: $a'x_3' = b'x_2'$, $x_2'^2$	$\mathbb{Z}_2^2$: $w_2'x_2$, $\frac{\mathcal{P}_2(x_2)}{2}$	$\mathbb{Z}_3^2$: $x_2'^2, x_2'z_2$
4 O	$\mathbb{Z}_2^4$: $x_2^2, w_1^4, w_1^2x_2, w_2^2$	$\mathbb{Z}_2^2$: w_1^4, w_2^2	$\mathbb{Z}_2^4$: $w_2'^2, w_1^4, w_1^2w_2', w_2^2$	$\mathbb{Z}_2^3$: w_1^4, w_2^2, $c_2 (\bmod 2)$	$\mathbb{Z}_2^8$: $w_1^4, w_2^2, a^4, a^2x_2, ax_3, x_2^2, w_1^2a^2, w_1^2x_2$	$\mathbb{Z}_2^2$: w_1^4, w_2^2	$\mathbb{Z}_2^7$: $w_1^4, w_2^2, x_2^2, w_2'^2, x_2w_1^2, w_2'w_1^2, w_2'x_2$	$\mathbb{Z}_2^3$: w_1^4, w_2^2, $c_2 (\bmod 2)$
4 Pin^+	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$: $q_s(x_2)$, η	$\mathbb{Z}_{16}$: η	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$: $q_s(w_2')$, η	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$: $c_2 (\bmod 2)$, η	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_{16}$: ax_3, $w_1ax_2 = a^2x_2 + ax_3$, $q_s(x_2)$, $w_1a\mathrm{ABK}$, η	$\mathbb{Z}_{16}$: η	$\mathbb{Z}_4^2 \times \mathbb{Z}_{16} \times \mathbb{Z}_2$: $q_s(w_2')$, $q_s(x_2)$, $\eta, w_2'x_2$	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$: $c_2 (\bmod 2)$, η
4 Pin^-	$\mathbb{Z}_2$: $w_1^2x_2$	0	$\mathbb{Z}_2$: $w_1^2w_2'$	$\mathbb{Z}_2$: $c_2 (\bmod 2)$	$\mathbb{Z}_2^3$: $w_1^2x_2, ax_3$, $w_1ax_2 = a^2x_2 + ax_3$	0	$\mathbb{Z}_2^3$: $w_1^2w_2'$, $w_1^2x_2, w_2'x_2$	$\mathbb{Z}_2$: $c_2 (\bmod 2)$

Table 7: TP_4.

$\mathrm{TP}_d(H \times -)$	$\mathrm{B}\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_3$	PSU(2)	PSU(3)	$\mathbb{Z}_2 \times \mathrm{B}\mathbb{Z}_2$	$\mathbb{Z}_3 \times \mathrm{B}\mathbb{Z}_3$	PSU(2) $\times$ $\mathrm{B}\mathbb{Z}_2$	PSU(3) $\times$ $\mathrm{B}\mathbb{Z}_3$
5 SO	$\mathbb{Z}_2^2$: $x_5 = x_2x_3$, w_2w_3	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^2$: w_2w_3, $w_2'w_3'$	$\mathbb{Z} \times \mathbb{Z}_2$: $\mathrm{CS}_5^{(\mathrm{PSU}(3))}$[13], w_2w_3	$\mathbb{Z}_2^6$: ax_2^2, a^5, $x_5 = x_2x_3, a^3x_2$, w_2w_3, aw_2^2	$\mathbb{Z}_2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_9$: w_2w_3, $a'b'x_2'$, $a'x_2'^2$, $\mathfrak{P}_3(b')$	$\mathbb{Z}_2^4$: $w_2'w_3', x_5 = x_2x_3$, $w_3'x_2 = w_2'x_3$, w_2w_3	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_3$: $\mathrm{CS}_5^{(\mathrm{PSU}(3))}$), w_2w_3, $z_2x_3' = -z_3x_2'$
5 Spin	0	0	0	$\mathbb{Z}$: $\frac{1}{2}\mathrm{CS}_5^{(\mathrm{PSU}(3))}$	$\mathbb{Z}_2$: a^3x_2	$\mathbb{Z}_3^2 \times \mathbb{Z}_9$: $a'b'x_2'$, $a'x_2'^2$, $\mathfrak{P}_3(b')$	$\mathbb{Z}_2$: $w_3'x_2 = w_2'x_3$	$\mathbb{Z} \times \mathbb{Z}_3$: $\frac{1}{2}\mathrm{CS}_5^{(\mathrm{PSU}(3))}$, $z_2x_3' = -z_3x_2'$
5 O	$\mathbb{Z}_2^4$: x_2x_3, $x_5 = (w_2 + w_1^2)x_3 = (w_3 + w_1^3)x_2$, $w_1^2x_3 = w_1^3x_2$, w_2w_3	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^3$: w_2w_3, $w_1^2w_3' = w_1^3w_2'$, $w_2'w_3'$.	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^{12}$: a^5, a^2x_3, $a^3x_2, a^3w_1^2$, ax_2^2, aw_1^4, $ax_2w_1^2, aw_2^2$, $x_2x_3, w_1^2x_3$, x_5, w_2w_3	$\mathbb{Z}_2$: w_2w_3	$\mathbb{Z}_2^8$: $w_2'w_3', x_2w_3'$, $w_1^2w_3', w_2'x_3$, $x_2x_3, w_1^2x_3$, x_5, w_2w_3	$\mathbb{Z}_2$: w_2w_3
5 Pin$^+$	$\mathbb{Z}_2^2$: x_2x_3, $x_5 = w_1^2x_3 = w_1^3x_2$	0	$\mathbb{Z}_2$: $w_1^2w_3' = w_2'w_3'$	0	$\mathbb{Z}_2^7$: w_1^4a, $a^5 = w_1^2a^3$, $w_1^2x_3 = x_5$, x_2x_3, $w_1^2ax_2 = ax_2^2 + a^2x_3$, $w_1ax_3 = a^2x_3$, a^3x_2	0	$\mathbb{Z}_2^5$: $w_1^2w_3' = w_2'w_3'$, $w_1^2x_3 = x_5$, $x_2x_3, w_3'x_2$, $w_1w_2'x_2 = w_2'x_3 + w_3'x_2$	0
5 Pin$^-$	$\mathbb{Z}_2$: x_2x_3	0	0	0	$\mathbb{Z}_2^5$: $w_1^2a^3, x_2x_3$, $w_1^2ax_2$, $w_1ax_3 = a^2x_3$, a^3x_2	0	$\mathbb{Z}_2^3$: $x_2x_3, w_3'x_2$, $w_1w_2'x_2 = w_2'x_3 + w_3'x_2$	0

Table 8: TP_5.

2 Background information

For more information, see [31, 52–54].

2.1 Cohomology theory

2.1.1 Cup product

Let X be a topological space, an n-simplex of X is a map $\sigma : \Delta^n \to X$ where

$$\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} | t_0 + t_1 + \cdots + t_n = 1, t_i \geq 0\}, \tag{2.1}$$

it is denoted by $[v_0, \dots, v_n]$ where v_i are vertices of Δ^n.

n-simplexes of X generates an abelian group $C_n(X)$, the elements of $C_n(X)$ are called n-chains. Δ^{n-1} embeds in Δ^n in the canonical way, define $\partial : C_n(X) \to C_{n-1}(X)$ by

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}. \tag{2.2}$$

It is easy to verify that $\partial^2 = 0$, so $(C_\bullet(X), \partial)$ is a chain complex.

Let G be an abelian group, let $C^n(X, G) := \mathrm{Hom}(C_n(X), G)$, the elements of $C^n(X, G)$ are called n-cochains with coefficients G. Define $\delta : C^n(X, G) \to C^{n+1}(X, G)$ by $\delta(\alpha)(\sigma) = \alpha(\partial(\sigma))$, then $\delta^2 = 0$, so $(C^\bullet(X, G), \delta)$ is a cochain complex.

$\mathrm{H}^n(X, G)$ is defined to be $\frac{\mathrm{Ker}\delta : C^n(X,G) \to C^{n+1}(X,G)}{\mathrm{Im}\delta : C^{n-1}(X,G) \to C^n(X,G)}$. It is an abelian group, called the n-th cohomology group of X with coefficients G, the elements of the abelian group $Z^n(X, G) := \mathrm{Ker}\delta : C^n(X, G) \to C^{n+1}(X, G)$ are called n-cocycles, the elements of $B^n(X, G) := \mathrm{Im}\delta : C^{n-1}(X, G) \to C^n(X, G)$ are called n-coboundaries.

By abusing the notation, we also use $[v_0, \dots, v_n]$ to denote an n-chain.

If G is additionally a ring R, then we can define a cup product such that $\mathrm{H}^*(X, R)$ is a graded ring. First we define the cup product of two cochains:

$$\begin{aligned} C^n(X, R) \times C^m(X, R) &\to C^{n+m}(X, R) \\ (\alpha, \beta) &\mapsto \alpha \cup \beta \end{aligned} \tag{2.3}$$

$$\alpha \cup \beta([v_0, \dots, v_{n+m}]) := \alpha([v_0, \dots, v_n]) \cdot \beta([v_n, \dots, v_{n+m}]) \tag{2.4}$$

where $\cdot$ is the multiplication in R.

The cup product satisfies

$$\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta + (-1)^n \alpha \cup (\delta\beta) \tag{2.5}$$

$\alpha \cup \beta$ is a cocycle if both α and β are cocycles. If both α and β are cocycles, then $\alpha \cup \beta$ is a coboundary if one of α and β is a coboundary. So the cup product is also an operation on cohomology groups $\cup : \mathrm{H}^n(X,R) \times \mathrm{H}^m(X,R) \to \mathrm{H}^{n+m}(X,R)$. The cup product of two cocycles satisfies

$$\alpha \cup \beta = (-1)^{nm} \beta \cup \alpha + \text{coboundary} \tag{2.6}$$

For the convenience of defining higher cup product, we use the notation $i \to j$ for the consecutive sequence from i to j

$$i \to j \equiv i, i+1, \cdots, j-1, j. \tag{2.7}$$

We also denote an n-chain by $(0 \to n)$. We use $\langle \alpha, \sigma \rangle$ to denote the value of $\alpha(\sigma)$ for n-cochain α and n-chain σ.

Let f_m be an m-cochain, h_n be an n-cochain, we define higher cup product $f_m \underset{k}{\cup} h_n$ which yields an $(m+n-k)$-cochain:

$$\begin{aligned} &\langle f_m \underset{k}{\cup} h_n, (0,1,\cdots,m+n-k) \rangle \\ = &\sum_{0 \le i_0 < \cdots < i_k \le n+m-k} (-1)^p \langle f_m, (0 \to i_0, i_1 \to i_2, \cdots) \rangle \times \langle h_n, (i_0 \to i_1, i_2 \to i_3, \cdots) \rangle, \end{aligned} \tag{2.8}$$

and $f_m \underset{k}{\cup} h_n = 0$ for $k > m$ or n or $k < 0$. Here $i \to j$ is the sequence $i, i+1, \cdots, j-1, j$, and p is the number of transpositions (it is not unique but its parity is unique) in the decomposition of the permutation to bring the sequence

$$0 \to i_0, i_1 \to i_2, \cdots; i_0 + 1 \to i_1 - 1, i_2 + 1 \to i_3 - 1, \cdots \tag{2.9}$$

to the sequence

$$0 \to m+n-k. \tag{2.10}$$

For example

$$\begin{aligned} &\langle f_m \underset{1}{\cup} h_n, (0,1,\cdots,m+n-1) \rangle = \sum_{i=0}^{m-1} (-1)^{(m-i)(n+1)} \times \\ &\langle f_m, (0 \to i, i+n \to m+n-1) \rangle \langle h_n, (i \to i+n) \rangle. \end{aligned} \tag{2.11}$$

We can see that $\underset{0}{\cup} = \cup$. Unlike cup product at $k = 0$, the higher cup product of two cocycles may not be a cocycle.

Steenrod studied the higher cup product of cochains and found a formula [55, Theorem 5.1]:

$$\delta(u \underset{i}{\cup} v) = (-1)^{p+q-i} u \underset{i-1}{\cup} v + (-1)^{pq+p+q} v \underset{i-1}{\cup} u + \delta u \underset{i}{\cup} v + (-1)^p u \underset{i}{\cup} \delta v \tag{2.12}$$

where u is a p-cochain, v is a q-cochain.

Also Steenrod defined Steenrod square using higher cup product:

$$\mathrm{Sq}^{n-k}(z_n) \equiv z_n \underset{k}{\cup} z_n \tag{2.13}$$

2.1.2 Universal coefficient theorem and Künneth formula

If X is a topological space, R is a principal ideal domain ($\mathbb{Z}$ or a field), G is an R-module, then the homology version of universal coefficient theorem is

$$\mathrm{H}_n(X,G) = \mathrm{H}_n(X,R) \otimes_R G \oplus \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(X,R),G). \tag{2.14}$$

The cohomology version of universal coefficient theorem is

$$\mathrm{H}^n(X,G) = \mathrm{Hom}_R(\mathrm{H}_n(X,R),G) \oplus \mathrm{Ext}_R^1(\mathrm{H}_{n-1}(X,R),G). \tag{2.15}$$

We will abbreviate $\mathrm{Tor}_1^{\mathbb{Z}}$ by Tor, $\mathrm{Ext}_{\mathbb{Z}}^1$ by Ext.

If X and X' are topological spaces, R is a principle ideal domain and G, G' are R-modules such that $\mathrm{Tor}_1^R(G,G') = 0$. We also require either
(1) $\mathrm{H}_n(X;\mathbb{Z})$ and $\mathrm{H}_n(X';\mathbb{Z})$ are finitely generated, or
(2) G' and $H_n(X';\mathbb{Z})$ are finitely generated.

The homology version of Künneth formula is

$$\begin{aligned} &\mathrm{H}_d(X \times X', G \otimes_R G') \\ \simeq &\Big[\oplus_{k=0}^d \mathrm{H}_k(X,G) \otimes_R \mathrm{H}_{d-k}(X',G') \Big] \oplus \Big[\oplus_{k=0}^{d-1} \mathrm{Tor}_1^R(\mathrm{H}^k(X,G), \mathrm{H}_{d-k-1}(X',G')) \Big]. \end{aligned} \tag{2.16}$$

The cohomology version of Künneth formula is

$$\begin{aligned} &\mathrm{H}^d(X \times X', G \otimes_R G') \\ \simeq &\Big[\oplus_{k=0}^d \mathrm{H}^k(X,G) \otimes_R \mathrm{H}^{d-k}(X',G') \Big] \oplus \Big[\oplus_{k=0}^{d+1} \mathrm{Tor}_1^R(\mathrm{H}^k(X,G), \mathrm{H}^{d-k+1}(X',G')) \Big]. \end{aligned} \tag{2.17}$$

Note that $\mathbb{Z}$ and $\mathbb{R}$ are principal ideal domains, while $\mathbb{R}/\mathbb{Z}$ is not. Also, $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are not finitely generate R-modules if $R = \mathbb{Z}$.

Special cases: 1. $R = G' = \mathbb{Z}$.

In this case, the condition $\mathrm{Tor}_1^R(G,G') = \mathrm{Tor}_1^{\mathbb{Z}}(G,\mathbb{Z}) = 0$ is always satisfied. G can be $\mathbb{R}/\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z}_n$ *etc* . So we have

$$\begin{aligned} &\mathrm{H}^d(X \times X', G) \\ \simeq &\Big[\oplus_{k=0}^d \mathrm{H}^k(X,G) \otimes_{\mathbb{Z}} \mathrm{H}^{d-k}(X';\mathbb{Z}) \Big] \oplus \Big[\oplus_{k=0}^{d+1} \mathrm{Tor}(\mathrm{H}^k(X,G), \mathrm{H}^{d-k+1}(X';\mathbb{Z})) \Big]. \end{aligned} \tag{2.18}$$

Take X to be the space of one point in (2.18), and use

$$\mathrm{H}^n(X,G)) = \begin{cases} G, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \tag{2.19}$$

to reduce (2.18) to

$$\mathrm{H}^d(X,G) \simeq \mathrm{H}^d(X;\mathbb{Z}) \otimes_{\mathbb{Z}} G \oplus \mathrm{Tor}(\mathrm{H}^{d+1}(X;\mathbb{Z}), G). \tag{2.20}$$

where X' is renamed as X. This is also called the universal coefficient theorem which can be used to calculate $\mathrm{H}^*(X,G)$ from $\mathrm{H}^*(X;\mathbb{Z})$ and the module G. Here $\mathrm{Tor} = \mathrm{Tor}_1^{\mathbb{Z}}$.

Homology version of (2.20) is just the universal coefficient theorem for homology with $R = \mathbb{Z}$.

2. $R = G = G' = \mathbb{F}$ is a field, $\mathrm{Tor}_1^R(G, G') = 0$.

$$\mathrm{H}^*(X \times X', \mathbb{F}) = \mathrm{H}^*(X, \mathbb{F}) \otimes \mathrm{H}^*(X', \mathbb{F}), \tag{2.21}$$

This is called the Künneth formula.

There is also a relative version of Künneth formula [52, Theorem 3.18]:

$$\tilde{\mathrm{H}}^*(X \wedge X', \mathbb{F}) = \tilde{\mathrm{H}}^*(X, \mathbb{F}) \otimes \tilde{\mathrm{H}}^*(X', \mathbb{F}). \tag{2.22}$$

Here $X \wedge X'$ is the smash product, $\tilde{\mathrm{H}}$ is the reduced cohomology.

2.2 Spectra

Definition 3. • A prespectrum $T_\bullet$ is a sequence $\{T_q\}_{q \in \mathbb{Z}^{\geq 0}}$ of pointed spaces and maps $s_q : \Sigma T_q \to T_{q+1}$.

• An Ω-prespectrum is a prespectrum $T_\bullet$ such that the adjoints $t_q : T_q \to \Omega T_{q+1}$ of the structure maps are weak homotopy equivalences.

• A spectrum is a prespectrum $T_\bullet$ such that the adjoints $t_q : T_q \to \Omega T_{q+1}$ of the structure maps are homeomorphisms.

Example 4. • Let X be a pointed space, $T_q = \Sigma^q X$ for $q \geq 0$, then $T_\bullet$ is a prespectrum.

• $T_q = S^q$, $T_\bullet$ is a prespectrum.

• Let G be an abelian group, $T_q = K(G,q)$ the Eilenberg-MacLane space, $T_\bullet$ is an Ω-prespectrum.

Spectrification: Let $T_\bullet$ be a prespectrum, define $(LT)_q$ to be the colimit of

$$T_q \xrightarrow{t_q} \Omega T_{q+1} \xrightarrow{\Omega t_{q+1}} \Omega^2 T_{q+2}.$$

Namely,

$$(LT)_q = \mathrm{colim}_{l \to \infty} \Omega^l T_{q+l},$$

then $(LT)_\bullet$ is a spectrum.

Example 5. • $T_q = S^q$, $(LT)_\bullet$ is a spectrum $\mathbb{S}$.

• Let G be an abelian group, $T_q = K(G, q)$ the Eilenberg-MacLane space, $(LT)_\bullet$ is a spectrum HG (the Eilenberg-MacLane spectrum).

Stable homotopy groups of spectra: Let $M_\bullet$ be a spectrum, define $\pi_d M_\bullet$ to be the colimit of

$$\pi_{d+n}M_n \xrightarrow{\pi_{d+n}t_n} \pi_{d+n}\Omega M_{n+1} \xrightarrow{\text{adjunction}} \pi_{d+n+1}M_{n+1}.$$

Namely,

$$\pi_d M_\bullet = \mathrm{colim}_{n\to\infty}\pi_{d+n}M_n.$$

Maps between spectra: If $M_\bullet, N_\bullet$ are two spectra, then for any integer k, the abelian group of homotopy classes of maps from $M_\bullet$ to $N_\bullet$ of degree $-k$: $[M_\bullet, N_\bullet]_{-k}$ is defined as follows: a map in $[M_\bullet, N_\bullet]_{-k}$ is a sequence of maps $M_n \to N_{n+k}$ such that the following diagram commutes

$$\begin{array}{ccc} \Sigma M_n & \longrightarrow & \Sigma N_{n+k} \\ \downarrow & & \downarrow \\ M_{n+1} & \longrightarrow & N_{n+k+1} \end{array} \tag{2.23}$$

where the columns are the structure maps of the spectra $M_\bullet$ and $N_\bullet$. If in addition the spectrum $N_\bullet$ is a ring spectrum, then the abelian groups $[M_\bullet, N_\bullet]_{-k}$ form a graded ring $[M_\bullet, N_\bullet]_{-*}$.

Example 6. $\pi_d M_\bullet = [\mathbb{S}, M_\bullet]_d$.

Cohomology rings of spectra:

Definition 7. A ring spectrum is a spectrum E along with a unit map $\eta : \mathbb{S} \to E$ and a multiplication map $\mu : E \wedge E \to E$.

Example 8. Let R be a ring, then the Eilenberg-MacLane spectrum HR is a ring spectrum.

The cohomology ring of a spectrum $M_\bullet$ with coefficients in R is defined to be $[M_\bullet, HR]_{-*}$.

2.3 Spectral sequences

In this paper, we use three kinds of spectral sequence: Adams spectral sequence, Atiyah-Hirzebruch spectral sequence, and Serre spectral sequence.

2.3.1 Adams spectral sequence

The Adams spectral sequence is a spectral sequence introduced by Adams in [56], it is of the form

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_p}^{s,t}(\mathrm{H}^*(Y, \mathbb{Z}_p), \mathbb{Z}_p) \Rightarrow \pi_{t-s}(Y)_p^\wedge \tag{2.24}$$

where Y is any spectrum. We consider $Y = MTH \wedge X_+$ and focus on $p = 2$ and $p = 3$.

We introduce the notions used in Adams spectral sequence:

p-completion: For any finitely generated abelian group G, $G_p^\wedge = \lim_n G/p^n G$ is the p-completion of G. If G is finite, then $G_p^\wedge$ is the Sylow p-subgroup of G. If $G = \mathbb{Z}$, $G_p^\wedge$ is the ring of p-adic integers.

Steenrod algebra: The mod p Steenrod algebra is $\mathcal{A}_p := [H\mathbb{Z}_p, H\mathbb{Z}_p]_{-*}$ where $H\mathbb{Z}_p$ is the mod p Eilenberg-MacLane spectrum. For any spectrum Y, the cohomology ring $\mathrm{H}^*(Y, \mathbb{Z}_p) = [Y, H\mathbb{Z}_p]_{-*}$ is an $\mathcal{A}_p$-module.

For $p = 2$, the generators of $\mathcal{A}_2$ are Steenrod squares Sq^n.

Definition 9 (Axioms)**.** For each $i \geq 0$, there is a natural transformation

$$\mathrm{Sq}^i : \mathrm{H}^n(-, \mathbb{Z}_2) \to \mathrm{H}^{n+i}(-, \mathbb{Z}_2)$$

such that
- $\mathrm{Sq}^0 = \mathrm{Id}$
- If $i > |x|$, then $\mathrm{Sq}^i x = 0$
- If $i = |x|$, then $\mathrm{Sq}^i x = x^2$
- (Cartan formula) $\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y)$
- (Adem relation) If $a < 2b$, then

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{c=0}^{[\frac{a}{2}]} \binom{b-c-1}{a-2c} \mathrm{Sq}^{a+b-c}\mathrm{Sq}^c$$

The subalgebra $\mathcal{A}_2(1)$ of $\mathcal{A}_2$ generated by Sq^1 and Sq^2 looks like Figure 1.

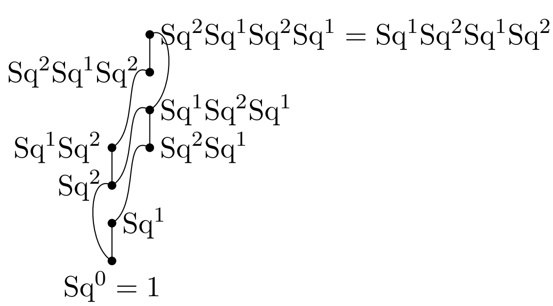


Figure 1: $\mathcal{A}_2(1)$

Each dot stands for a $\mathbb{Z}_2$, all relations are from Adem relations (2.67).

For odd primes p, the generators of $\mathcal{A}_p$ are the Bockstein homomorphism $\beta_{(p,p)}$ and Steenrod powers P_p^n.

Ext functor: Let $R = \mathcal{A}_p$ or $\mathcal{A}_2(1)$. $\mathrm{Ext}_R^{s,t}$ is the internal degree t part of the s-th derived functor of Hom_R^*.

In general, we can find a projective R-resolution $P_\bullet$ of L to compute $\mathrm{Ext}_R^i(L,\mathbb{Z}_p)$, $\mathrm{Ext}_R^i(L,\mathbb{Z}_p) = \mathrm{H}^i(\mathrm{Hom}_R(P_\bullet,\mathbb{Z}_p))$ (the i-th cohomology of the chain complex $\mathrm{Hom}_R(P_\bullet,\mathbb{Z}_p)$).

In Adams chart, the horizontal axis is degree $t-s$ and the vertical axis is degree s. The differential $d_r^{s,t}: E_r^{s,t} \to E_r^{s+r,t+r-1}$ is an arrow starting at the bidegree $(t-s,s)$ with direction $(-1,r)$. $E_{r+1}^{s,t} := \frac{\mathrm{Ker} d_r^{s,t}}{\mathrm{Im} d_r^{s-r,t-r+1}}$ for $r \geq 2$. There exists N such that $E_{N+k} = E_N$ for $k > 0$, denote $E_\infty := E_N$.

We explain how to read the result from the Adams chart: In the E_∞ page, one dot indicates a $\mathbb{Z}_p$, an vertical line connecting n dots indicates a $\mathbb{Z}_{p^n}$, when $n=\infty$, the line indicates a $\mathbb{Z}$.

In the $H = \mathrm{O}$ cases, $M\mathrm{O}$ is the wedge sum of suspensions of the Eilenberg-MacLane spectrum $H\mathbb{Z}_2$, $\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)$ is the direct sum of suspensions of the Steenrod algebra $\mathcal{A}_2$.

$\mathrm{H}^*(M\mathrm{O} \wedge X_+,\mathbb{Z}_2) = \mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2) \otimes \mathrm{H}^*(X,\mathbb{Z}_2)$ is also the direct sum of suspensions of the Steenrod algebra $\mathcal{A}_2$. We have used the Künneth formula (2.22). Let $L = \mathrm{H}^*(M\mathrm{O} \wedge X_+,\mathbb{Z}_2)$, then $P_0 = L$, $P_s = 0$ for $s > 0$ gives a projective $\mathcal{A}_2$-resolution of L.

Since

$$\mathrm{Ext}_{\mathcal{A}_p}^{s,t}(\Sigma^r\mathcal{A}_p,\mathbb{Z}_p) = \begin{cases} \mathrm{Hom}_{\mathcal{A}_p}^t(\Sigma^r\mathcal{A}_p,\mathbb{Z}_p) = \mathbb{Z}_p & \text{if } t=r, s=0 \\ 0 & \text{else} \end{cases}, \tag{2.25}$$

all dots are concentrated in $s=0$ in the Adams chart of $\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O} \wedge X_+,\mathbb{Z}_2),\mathbb{Z}_2)$, there are no differentials, $E_2 = E_\infty$, $\Omega_d^{\mathrm{O}}(X)$ is a $\mathbb{Z}_2$-vector space.

In the $H = \mathrm{SO}$ cases, the localization of $M\mathrm{SO}$ at the prime 2 is

$$M\mathrm{SO}_{(2)} = H\mathbb{Z}_{(2)} \vee \Sigma^4 H\mathbb{Z}_{(2)} \vee \Sigma^5 H\mathbb{Z}_2 \vee \cdots \tag{2.26}$$

where $H\mathbb{Z}$ is the Eilenberg-MacLane spectrum and $\mathrm{H}^*(H\mathbb{Z},\mathbb{Z}_2) = \mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1$.

$$\cdots \longrightarrow \Sigma^3\mathcal{A}_2 \longrightarrow \Sigma^2\mathcal{A}_2 \longrightarrow \Sigma\mathcal{A}_2 \longrightarrow \mathcal{A}_2 \longrightarrow \mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \tag{2.27}$$

is an $\mathcal{A}_2$-resolution (denoted by $P_\bullet$) where the differentials d_1 are induced by Sq^1.

When X is a point, the Adams chart of $\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2),\mathbb{Z}_2)$ is shown in Figure 2. For general X, $P_\bullet \otimes \mathrm{H}^*(X,\mathbb{Z}_2)$ is a projective $\mathcal{A}_2$-resolution of $\mathrm{H}^*(H\mathbb{Z},\mathbb{Z}_2) \otimes \mathrm{H}^*(X,\mathbb{Z}_2)$ (since $P_\bullet$ is actually a free $\mathcal{A}_2$-resolution), the differentials d_1 are induced by Sq^1.

The localization of $M\mathrm{SO}$ at the prime 3 is the wedge sum of suspensions of the Brown-Peterson spectrum BP ($M\mathrm{SO}_{(3)} = BP \vee \Sigma^8 BP \vee \cdots$) and $\mathrm{H}^*(BP,\mathbb{Z}_3) = \mathcal{A}_3/(\beta_{(3,3)})$ where $(\beta_{(3,3)})$ is the two-sided ideal generated by $\beta_{(3,3)}$.

$$\cdots \longrightarrow \Sigma^2\mathcal{A}_3 \oplus \Sigma^6\mathcal{A}_3 \oplus \cdots \longrightarrow \Sigma\mathcal{A}_3 \oplus \Sigma^5\mathcal{A}_3 \oplus \cdots \longrightarrow \mathcal{A}_3 \longrightarrow \mathcal{A}_3/(\beta_{(3,3)}) \tag{2.28}$$

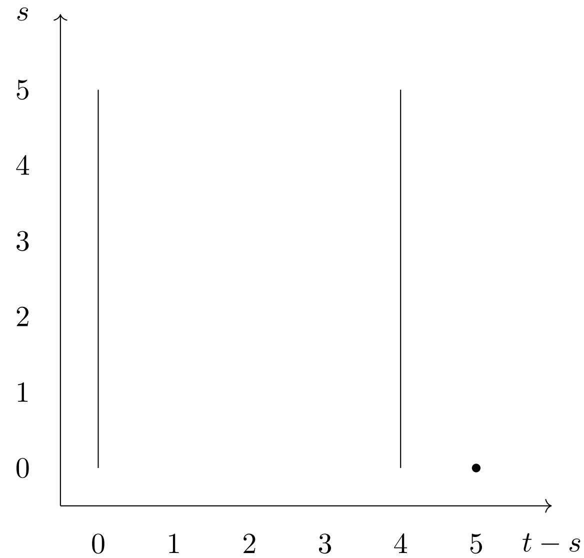


Figure 2: Adams chart of $\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2),\mathbb{Z}_2)$

is an $\mathcal{A}_3$-resolution of $\mathcal{A}_3/(\beta_{(3,3)})$ (denoted by $P'_\bullet$) where the differentials d_1 are induced by $\beta_{(3,3)}$.

When X is a point, the Adams chart of $\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_3),\mathbb{Z}_3)$ is shown in Figure 3. For general X, $P'_\bullet \otimes \mathrm{H}^*(X,\mathbb{Z}_3)$ is a projective $\mathcal{A}_3$-resolution of $\mathrm{H}^*(BP,\mathbb{Z}_3)\otimes \mathrm{H}^*(X,\mathbb{Z}_3)$ (since $P'_\bullet$ is actually a free $\mathcal{A}_3$-resolution), the differentials d_1 are induced by $\beta_{(3,3)}$.

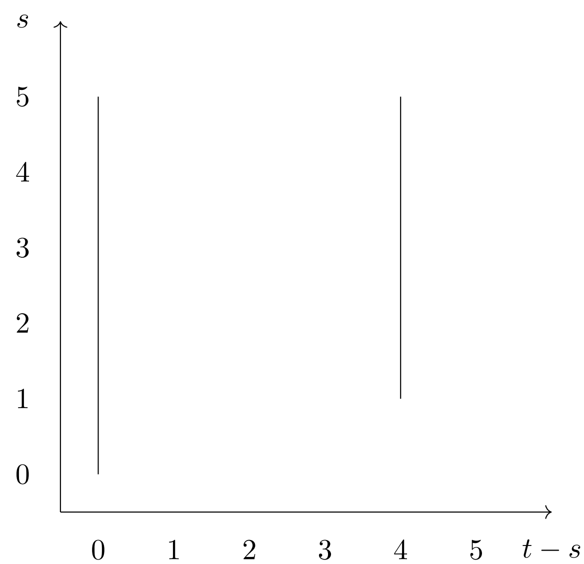


Figure 3: Adams chart of $\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_3),\mathbb{Z}_3)$

There may be differentials d_n corresponding to the Bockstein homomorphism $\beta_{(p,p^n)}$ [57] for both $p=2$ and $p=3$. See 2.5 for the definition of Bockstein homomorphisms. Since $M\mathrm{SO}_{(3)} = M\mathrm{Spin}_{(3)}$, $\Omega_d^{\mathrm{SO}}(X)_3^\wedge = \Omega_d^{\mathrm{Spin}}(X)_3^\wedge$.

In the $H = \text{Spin}/\text{Pin}^{\pm}$ cases, since the mod 2 cohomology of the Thom spectrum $M\text{Spin}$ is

$$\mathrm{H}^*(M\text{Spin}, \mathbb{Z}_2) = \mathcal{A}_2 \otimes_{\mathcal{A}_2(1)} \{\mathbb{Z}_2 \oplus M\} \tag{2.29}$$

where M is a graded $\mathcal{A}_2(1)$-module with the degree i homogeneous part $M_i = 0$ for $i < 8$.

Theorem 10 (Change of rings/Frobenius reciprocity)**.**

$$\text{Ext}^{s,t}_{\mathcal{A}_2}(\mathcal{A}_2 \otimes_{\mathcal{A}_2(1)} L, \mathbb{Z}_2) \cong \text{Ext}^{s,t}_{\mathcal{A}_2(1)}(L, \mathbb{Z}_2)$$

When we compute $\Omega^H_d(X)^\wedge_2$, we are reduced to compute $\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(L, \mathbb{Z}_2)$ for $t - s < 8$, where L is some $\mathcal{A}_2(1)$-module (our cases are some mod 2 cohomology $\mathrm{H}^*(-, \mathbb{Z}_2)$).

Example 1: $L = \mathcal{A}_2(1)$,

$$\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(\mathcal{A}_2(1), \mathbb{Z}_2) = \begin{cases} \text{Hom}_{\mathcal{A}_2(1)}(\mathcal{A}_2(1), \mathbb{Z}_2) = \mathbb{Z}_2 & \text{if } t = s = 0 \\ 0 & \text{else} \end{cases} \tag{2.30}$$

Example 2: $L = \mathbb{Z}_2$, the $\mathcal{A}_2(1)$-resolution of L is

$$\cdots \to \Sigma^3\mathcal{A}_2(1) \oplus \Sigma^7\mathcal{A}_2(1) \to \Sigma^2\mathcal{A}_2(1) \oplus \Sigma^4\mathcal{A}_2(1) \to \Sigma\mathcal{A}_2(1) \oplus \Sigma^2\mathcal{A}_2(1) \to \mathcal{A}_2(1) \to \mathbb{Z}_2. \tag{2.31}$$

The Adams chart looks like Figure 4.

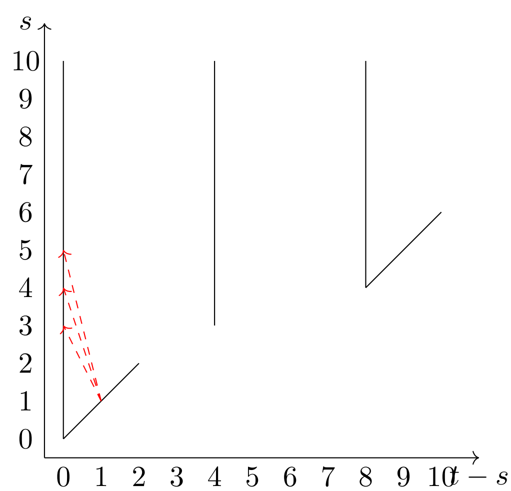


Figure 4: Adams chart of $\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(\mathbb{Z}_2, \mathbb{Z}_2)$. The dashed arrows indicate the possible differentials.

The only possible differentials are $d_r(h_1) = h_0^{r+1}$ where $h_0 \in \text{Ext}^{1,1}_{\mathcal{A}_2(1)}(\mathbb{Z}_2\mathbb{Z}_2)$, $h_1 \in \text{Ext}^{1,2}_{\mathcal{A}_2(1)}(\mathbb{Z}_2\mathbb{Z}_2)$. If there were such a differential d_r for $r \geq 2$, then since $h_0h_1 = 0$, $0 = d_r(h_0h_1) = h_0^{r+2}$ which is not true. Hence $E_2 = E_\infty$.

This is in fact real Bott periodicity ($\pi_{i+8}ko = \pi_i ko$, $\mathrm{H}^*(ko, \mathbb{Z}_2) = \mathcal{A}_2 \otimes_{\mathcal{A}_2(1)} \mathbb{Z}_2$).

Our computation is based on the following fact:

Lemma 11. Given a short exact sequence of $\mathcal{A}_2(1)$-modules

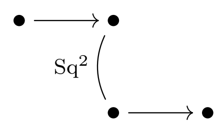
$$0 \to L_1 \to L_2 \to L_3 \to 0, \tag{2.32}$$

then for any t, there is a long exact sequence

$$\begin{aligned} &\cdots \to \mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_3, \mathbb{Z}_2) \to \mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_2, \mathbb{Z}_2) \to \mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_1, \mathbb{Z}_2) \\ &\xrightarrow{d_1} \mathrm{Ext}^{s+1,t}_{\mathcal{A}_2(1)}(L_3, \mathbb{Z}_2) \to \mathrm{Ext}^{s+1,t}_{\mathcal{A}_2(1)}(L_2, \mathbb{Z}_2) \to \cdots \end{aligned} \tag{2.33}$$

After using this fact repeatedly, we obtain the E_2 page.

Example 3:



(2.34)

is a short exact sequence where the left dot is L_1, the middle part is L_2, the right dot is L_3.

The Adams chart looks like Figure 5.

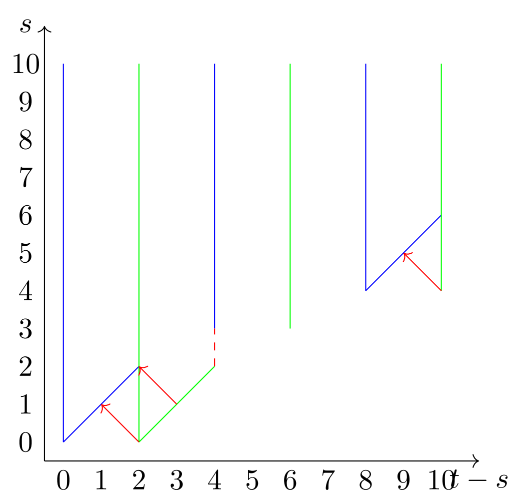
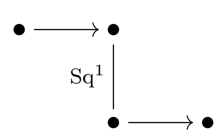


Figure 5: Adams chart of $\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_2, \mathbb{Z}_2)$. The arrows indicate the differential d_1, the dashed line indicates the extension.

Example 4:



(2.35)

is a short exact sequence where the left dot is L_1', the middle part is L_2', the right dot is L_3'.

The Adams chart looks like Figure 6.

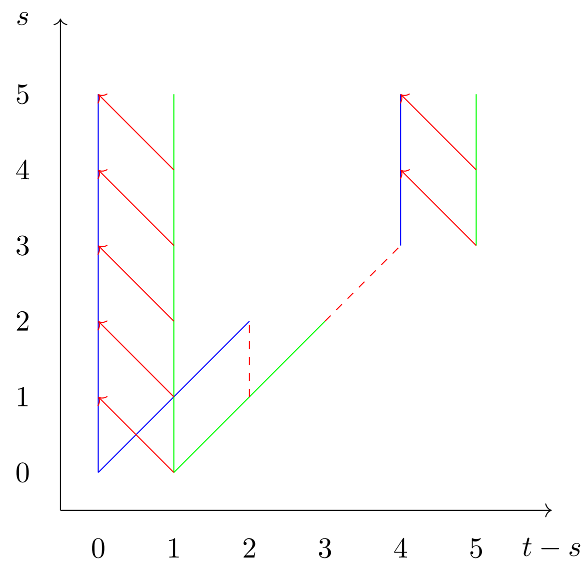


Figure 6: Adams chart of $\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_2', \mathbb{Z}_2)$. The arrows indicate the differential d_1, the dashed line indicates the extension.

2.3.2 Serre spectral sequence

Given a fibration $F \to E \to B$, the Serre spectral sequence is the following:

$$E_2^{p,q} = \mathrm{H}^p(B, H^q(F, \mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(E, \mathbb{Z}) \tag{2.36}$$

This can be used in computing the integral cohomology group of the total space of a nontrivial fibration.

There is also a homology version:

$$E^2_{p,q} = \mathrm{H}_p(B, H_q(F, \mathbb{Z})) \Rightarrow \mathrm{H}_{p+q}(E, \mathbb{Z}) \tag{2.37}$$

2.3.3 Atiyah-Hirzebruch spectral sequence

The Atiyah-Hirzebruch spectral sequence can be viewed as a generalization of the Serre spectral sequence. Given a fibration $F \to E \to B$, the Atiyah-Hirzebruch spectral sequence is the following:

$$E^2_{p,q} = \mathrm{H}_p(B, h_q(F, \mathbb{Z})) \Rightarrow h_{p+q}(E, \mathbb{Z}) \tag{2.38}$$

where h_* is an extraordinary homology theory. For example, h_* can be the bordism theory Ω^H_*. In particular, if the fiber F is a point, then the Atiyah-Hirzebruch spectral sequence is of the form:

$$\mathrm{H}_p(X, \Omega^H_q) \Rightarrow \Omega^H_{p+q}(X) \tag{2.39}$$

2.4 Characteristic classes

2.4.1 Introduction to characteristic classes

Characteristic classes are cohomology classes of the base space of a vector bundle. Stiefel-Whitney classes are defined for real vector bundles, Chern classes are defined for complex vector bundles, Pontryagin classes are defined for real vector bundles. All characteristic classes are natural with respect to bundle maps. Characteristic classes of a principal bundle are defined to be the characteristic classes of the associated vector bundle of the principal bundle.

Given a real vector bundle $V \to M$ and a complex vector bundle $E \to M$, the i-th Stiefel-Whitney class of V is $w_i(V) \in \mathrm{H}^i(M, \mathbb{Z}_2)$, the i-th Chern class of E is $c_i(E) \in \mathrm{H}^{2i}(M, \mathbb{Z})$, the i-th Pontryagin class of V is $p_i(V) \in \mathrm{H}^{4i}(M, \mathbb{Z})$.

Pontryagin classes are closely related to Chern classes via complexification:

$$p_i(V) = (-1)^i c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C}) \in \mathrm{H}^{4i}(M, \mathbb{Z}) \tag{2.40}$$

where $V \otimes_{\mathbb{R}} \mathbb{C} \to M$ is the complexification of the real vector bundle $V \to M$.

The relation between Pontryagin classes and Stiefel-Whitney classes is

$$p_i(V) = w_{2i}(V)^2 \mod 2. \tag{2.41}$$

For a manifold M, the integrals over M of characteristic classes of a vector bundle over M (the pairing of the characteristic classes with the fundamental class of M) are called characteristic numbers.

Let E_n be the universal n-bundle over $\mathrm{BO}(n)$, the colimit of $E_n - n$ is a virtual bundle E (of dimension 0) over BO, the pullback of E along the map $g : M \to \mathrm{BO}$ given by the O-structure on M is just $TM - d$ where M is a d-manifold and TM is the tangent bundle of M. By the naturality of characteristic classes, the pullback of the characteristic classes of E is the characteristic classes of TM.

Chern-Simons form: By Chern-Weil theory, Chern classes (and Pontryagin classes) can also be defined as a closed differential form (in de Rham cohomology). By Poincaré Lemma, they are exact locally:

$$c_n = \mathrm{dCS}_{2n-1} \tag{2.42}$$

where d is the exterior differential operator, CS_{2n-1} is called the Chern-Simons $2n-1$-form.

Whitney sum formula: Let $w(V) = 1 + w_1(V) + w_2(V) + \cdots \in \mathrm{H}^*(M, \mathbb{Z}_2)$ be the total Stiefel-Whitney class, $c(E) = 1 + c_1(E) + c_2(E) + \cdots \in \mathrm{H}^*(M, \mathbb{Z})$ be the total Chern class, $p(V) = 1 + p_1(V) + p_2(V) + \cdots \in \mathrm{H}^*(M, \mathbb{Z})$ be the total Pontryagin class, then

$$w(V \oplus V') = w(V)w(V'), \tag{2.43}$$

$$c(E \oplus E') = c(E)c(E'), \tag{2.44}$$

$$2p(V \oplus V') = 2p(V)p(V'). \tag{2.45}$$

That is, the total Stiefel-Whitney class and the total Chern class are multiplicative with respect to Whitney sum of vector bundles, the total Pontryagin class is multiplicative modulo 2-torsion with respect to Whitney sum of vector bundles.

2.4.2 Wu formulas

The total Stiefel-Whitney class $w = 1 + w_1 + w_2 + \cdots$ is related to the total Wu class $u = 1 + u_1 + u_2 + \cdots$ through the total Steenrod square:

$$w = \mathrm{Sq}(u), \quad \mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots . \tag{2.46}$$

Therefore, $w_n = \sum_{i=0}^{n} \mathrm{Sq}^i(u_{n-i})$. The Steenrod squares satisfy:

$$\mathrm{Sq}^i(x_j) = 0, \; i > j, \quad \mathrm{Sq}^j(x_j) = x_j x_j, \quad \mathrm{Sq}^0 = 1, \tag{2.47}$$

for any $x_j \in \mathrm{H}^j(M^d; \mathbb{Z}_2)$. Thus

$$u_n = w_n + \sum_{i=1, 2i \le n} \mathrm{Sq}^i(u_{n-i}). \tag{2.48}$$

This allows us to compute u_n iteratively, using Wu formula

$$\begin{aligned} &\mathrm{Sq}^i(w_j) = 0, \quad i > j, \qquad \mathrm{Sq}^i(w_i) = w_i w_i, \\ &\mathrm{Sq}^i(w_j) = w_i w_j + \sum_{k=1}^{i} \binom{j-i-1+k}{k} w_{i-k} w_{j+k}, \quad i < j, \end{aligned} \tag{2.49}$$

and the Steenrod relation

$$\mathrm{Sq}^n(xy) = \sum_{i=0}^{n} \mathrm{Sq}^i(x) \mathrm{Sq}^{n-i}(y). \tag{2.50}$$

We find

$$\begin{aligned} &u_0 = 1, \qquad u_1 = w_1, \qquad u_2 = w_1^2 + w_2, \\ &u_3 = w_1 w_2, \qquad u_4 = w_1^4 + w_2^2 + w_1 w_3 + w_4, \\ &u_5 = w_1^3 w_2 + w_1 w_2^2 + w_1^2 w_3 + w_1 w_4. \end{aligned} \tag{2.51}$$

On the tangent bundle of M^d, the corresponding Wu class and the Steenrod square satisfy

$$\mathrm{Sq}^{d-j}(x_j) = u_{d-j} x_j, \text{ for any } x_j \in \mathrm{H}^j(M^d; \mathbb{Z}_2). \tag{2.52}$$

This is also called Wu formula.

2.5 Bockstein homomorphisms

In general, given a chain complex $C_\bullet$ and a short exact sequence of abelian groups:

$$0 \to A' \to A \to A'' \to 0, \tag{2.53}$$

we have a short exact sequence of cochain complexes:

$$0 \to \mathrm{Hom}(C_\bullet, A') \to \mathrm{Hom}(C_\bullet, A) \to \mathrm{Hom}(C_\bullet, A'') \to 0. \tag{2.54}$$

Hence we obtain a long exact sequence of cohomology groups:

$$\cdots \to \mathrm{H}^n(C_\bullet, A') \to \mathrm{H}^n(C_\bullet, A) \to \mathrm{H}^n(C_\bullet, A'') \xrightarrow{\partial} \mathrm{H}^{n+1}(C_\bullet, A') \to \cdots, \tag{2.55}$$

the connecting homomorphism ∂ is called Bockstein homomorphism.

For example, $\beta_{(n,m)} : \mathrm{H}^*(-, \mathbb{Z}_m) \to \mathrm{H}^{*+1}(-, \mathbb{Z}_n)$ is the Bockstein homomorphism associated to the extension $\mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_{nm} \to \mathbb{Z}_m$.

Let $\rho_{(nm,m)} : \mathrm{H}^*(-, \mathbb{Z}_{nm}) \to \mathrm{H}^*(-, \mathbb{Z}_m)$ be the mod m reduction map, then $\beta_{(n,m)}\rho_{(nm,m)} = 0$ by the long exact sequence. In particular, $\beta_{(2,2)}\rho_{(4,2)} = 0$.

Relations between the Bockstein homomorphisms: If we have a chain complex $C_\bullet$ and a commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \end{array}, \tag{2.56}$$

then we have a commutative diagram of cochain complexes with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(C_\bullet, C') & \longrightarrow & \mathrm{Hom}(C_\bullet, C) & \longrightarrow & \mathrm{Hom}(C_\bullet, C'') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(C_\bullet, A') & \longrightarrow & \mathrm{Hom}(C_\bullet, A) & \longrightarrow & \mathrm{Hom}(C_\bullet, A'') & \longrightarrow & 0 \end{array}, \tag{2.57}$$

By the naturality of the connecting homomorphism [58, Theorem 6.13], we have a commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathrm{H}^n(C_\bullet, C') & \longrightarrow & \mathrm{H}^n(C_\bullet, C) & \longrightarrow & \mathrm{H}^n(C_\bullet, C'') & \xrightarrow{\partial} & \mathrm{H}^{n+1}(C_\bullet, C') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{H}^n(C_\bullet, A') & \longrightarrow & \mathrm{H}^n(C_\bullet, A) & \longrightarrow & \mathrm{H}^n(C_\bullet, A'') & \xrightarrow{\partial'} & \mathrm{H}^{n+1}(C_\bullet, A') & \longrightarrow & \cdots \end{array} \tag{2.58}$$

There are commutative diagrams:

$$\begin{array}{ccccc} \mathbb{Z}_n & \xrightarrow{\cdot m} & \mathbb{Z}_{nm} & \xrightarrow{\text{mod } m} & \mathbb{Z}_m \\ \| & & \downarrow{\cdot k} & & \downarrow{\cdot k} \\ \mathbb{Z}_n & \xrightarrow{\cdot km} & \mathbb{Z}_{knm} & \xrightarrow{\text{mod } km} & \mathbb{Z}_{km} \end{array} \tag{2.59}$$

$$\begin{CD} \mathbb{Z}_{kn} @>{\cdot m}>> \mathbb{Z}_{knm} @>{\bmod m}>> \mathbb{Z}_m \\ @V{\bmod n}VV @V{\bmod nm}VV @| \\ \mathbb{Z}_n @>{\cdot m}>> \mathbb{Z}_{nm} @>{\bmod m}>> \mathbb{Z}_{km} \end{CD} \tag{2.60}$$

By (2.58), we have the following commutative diagrams:

$$\begin{CD} \mathrm{H}^*(-,\mathbb{Z}_m) @>{\beta_{(n,m)}}>> \mathrm{H}^{*+1}(-,\mathbb{Z}_n) \\ @V{\cdot k}VV @| \\ \mathrm{H}^*(-,\mathbb{Z}_{km}) @>{\beta_{(n,km)}}>> \mathrm{H}^{*+1}(-,\mathbb{Z}_n) \end{CD} \tag{2.61}$$

$$\begin{CD} \mathrm{H}^*(-,\mathbb{Z}_m) @>{\beta_{(kn,m)}}>> \mathrm{H}^{*+1}(-,\mathbb{Z}_{kn}) \\ @| @VV{\bmod n}V \\ \mathrm{H}^*(-,\mathbb{Z}_m) @>{\beta_{(n,m)}}>> \mathrm{H}^{*+1}(-,\mathbb{Z}_n) \end{CD} \tag{2.62}$$

Hence we have

$$\beta_{(n,m)} = \beta_{(n,km)} \cdot k, \tag{2.63}$$

and

$$\rho_{(kn,n)}\beta_{(kn,m)} = \beta_{(n,m)}. \tag{2.64}$$

By definition,

$$\beta_{(2,2^n)} = \frac{1}{2^n}\delta \mod 2 \tag{2.65}$$

where δ is the coboundary map.

Moreover, $\mathrm{Sq}^1 = \beta_{(2,2)}$.

By (2.64), $\beta_{(2,4)} = \rho_{(4,2)}\beta_{(4,4)}$, thus $\beta_{(2,2)}\beta_{(2,4)} = \beta_{(2,2)}\rho_{(4,2)}\beta_{(4,4)} = 0$.

Similarly, $\beta_{(2,8)} = \rho_{(4,2)}\beta_{(4,8)}$, thus $\beta_{(2,2)}\beta_{(2,8)} = \beta_{(2,2)}\rho_{(4,2)}\beta_{(4,8)} = 0$, etc.

Combining this with the Adem relation $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$, we obtain the important formula:

$$\mathrm{Sq}^1\beta_{(2,2^n)} = 0 \tag{2.66}$$

2.6 Useful fomulas

Adem relations:

$$\mathrm{Sq}^a\mathrm{Sq}^b=\sum_{j=0}^{[a/2]}\binom{b-1-j}{a-2j}\mathrm{Sq}^{a+b-j}\mathrm{Sq}^j \tag{2.67}$$

for $0<a<2b$. In particular, we have $\mathrm{Sq}^1\mathrm{Sq}^1=0$, $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1=\mathrm{Sq}^2\mathrm{Sq}^2$.

Recall that

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2[a] \tag{2.68}$$

where a is the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)$.

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2[x_2,x_3,x_5,x_9,\dots] \tag{2.69}$$

where x_2 is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$, $x_3=\mathrm{Sq}^1x_2$, $x_5=\mathrm{Sq}^2x_3$, $x_9=\mathrm{Sq}^4x_5$, etc.

$$\mathrm{H}^*(\mathrm{BPSU}(2),\mathbb{Z}_2)=\mathbb{Z}_2[w_2',w_3'] \tag{2.70}$$

where w_i' is the i-th Stiefel-Whitney class of the universal $\mathrm{PSU}(2)=\mathrm{SO}(3)$ bundle.

Combining (2.52) and (2.50), we have the following useful formulas in the presentation of cobordsim invariants:

$$\begin{aligned}
a^2=\mathrm{Sq}^1a &= w_1a \text{ in 2d} && (2.71)\\
x_3=\mathrm{Sq}^1x_2 &= w_1x_2 \text{ in 3d} && (2.72)\\
w_3'=\mathrm{Sq}^1w_2' &= w_1w_2' \text{ in 3d} && (2.73)\\
\mathrm{Sq}^1(ax_2) &= a^2x_2+ax_3=w_1ax_2 \text{ in 4d} && (2.74)\\
\mathrm{Sq}^2(ax_2) &= ax_2^2+a^2x_3=(w_2+w_1^2)ax_2 \text{ in 5d} && (2.75)\\
x_5=\mathrm{Sq}^2x_3 &= (w_2+w_1^2)x_3 \text{ in 5d} && (2.76)\\
w_2'w_3'=\mathrm{Sq}^2(w_3') &= (w_2+w_1^2)w_3' \text{ in 5d} && (2.77)\\
\mathrm{Sq}^1(w_2x_2) &= (w_1w_2+w_3)x_2+w_2x_3=w_1w_2x_2\Rightarrow w_3x_2=w_2x_3 \text{ in 5d} && (2.78)\\
\mathrm{Sq}^1(w_1^2x_2) &= w_1^2x_3=w_1^3x_2 \text{ in 5d} && (2.79)\\
\mathrm{Sq}^3x_2 &= w_1w_2x_2=0 \text{ in 5d} && (2.80)\\
\mathrm{Sq}^1(w_2w_2') &= (w_1w_2+w_3)w_2'+w_2w_3'=w_1w_2w_2'\Rightarrow w_3w_2'=w_2w_3' \text{ in 5d} && (2.81)\\
\mathrm{Sq}^1(w_1^2w_2') &= w_1^2w_3'=w_1^3w_2' \text{ in 5d} && (2.82)\\
\mathrm{Sq}^3w_2' &= w_1w_2w_2'=0 \text{ in 5d} && (2.83)\\
\mathrm{Sq}^1(x_2^2) &= w_1x_2^2=2x_2x_3=0 \text{ in 5d} && (2.84)\\
\mathrm{Sq}^1(w_2'^2) &= w_1w_2'^2=2w_2'w_3'=0 \text{ in 5d} && (2.85)\\
\mathrm{Sq}^1(w_2'x_2) &= w_3'x_2+w_2'x_3=w_1w_2'x_2 \text{ in 5d} && (2.86)
\end{aligned}$$

where w_i is the i-th Stiefel-Whitney class of the tangent bundle of M, all cohomology classes are pulled back to M along the maps given in the definition of cobordism groups.

3 Warm-Up Examples

3.1 Perturbative chiral anomalies in even dd and associated Chern-Simons $(d+1)$-form theories — SO- and Spin- Cobordism groups of BU(1)

3.1.1 Perturbative bosonic/fermionic anomaly in an even dd and U(1) SPTs in an odd $(d+1)$d

We start from a warming-up example familiar to most physicists and quantum field theorists: the perturbative anomalies that can be captured by a 1-loop calculation via Feynman-Dyson diagrams involved with a U(1) group. Of course, our discussion on the U(1) group can be generalized to any compact semi-simple Lie group such as SU(N), although we focus mostly on U(1) in this section. We will consider a Dirac fermion theory in any even dimensional spacetime, denoted as dd with d as the even integer (say $d = 2, 4, 6, 8, 10, \dots$). The Dirac fermion Ψ (or a complex Dirac spinor) is in a $2^{[d/2]}$-dimensional spinor representation of Spin(1,$d-1$) (or Spin(d) in the Euclidean signature, where $\mathrm{Spin}(d)/\mathbb{Z}_2^F = \mathrm{SO}(d)$, with the continuous spacetime rotational symmetry SO(d) and the fermion parity $\mathbb{Z}_2^F$ symmetry acts on any fermion $\Psi \to -\Psi$). The Dirac fermion Ψ can be coupled to non-dynamical U(1) or dynamical U(1) gauge fields, as Model (1) and Model (2) below respectively.

Model (1): The U(1) is treated as a U(1) global internal symmetry for the 't Hooft anomaly. The so-called path integral or partition function $\mathbf{Z}$ of this Dirac fermion theory is defined as a functional integral (here in Minkowski signature):

$$\mathbf{Z}[A] := \int [D\bar{\Psi}][D\Psi] \exp\big(+\mathrm{i} S_{M,\mathrm{Dirac}}\big) \equiv \int [D\bar{\Psi}][D\Psi] \exp\Big(+\mathrm{i}\int_{M^d} \mathrm{d}^d x (\bar{\Psi}(\mathrm{i}\not{D}_A)\Psi)\Big), \quad (3.1)$$

where $\not{D}_A$ is the Dirac operator endorsed with the Feynman slash notation, defined as:

$$\not{D}_A := \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu - \mathrm{i} g A_\mu). \quad (3.2)$$

The g is the coupling constant for the non-dynamical 1-form U(1) gauge field $A := A_\mu \mathrm{d}x^\mu$. The γ^μ with $\mu = 0, 1, \dots, d-1$ are so-called gamma matrices, satisfying the Clifford algebra $\mathrm{C}\ell_{1,\mathrm{d}-1}(\mathbb{R})$ under the anti-commutator constraint:

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{I}_{2^{[d/2]}} \quad (3.3)$$

The standard Dirac matrices correspond to $d = 2^[d/2] = 4$. The $\eta^{\mu\nu}$ is the Minkowski metric

$$\eta^{\mu\nu} := \mathrm{diag}(+, -, -, \dots, -),$$

with one $+$ sign and $(d-1)$ $-$ sign along the diagonal. The hermitian chiral matrix $\gamma^{\mathrm{Chiral}} \equiv \gamma^{\mathrm{FIVE}}$ can be defined for even d dimensions

$$\gamma^{\mathrm{Chiral}} \equiv \gamma^{\mathrm{FIVE}} := \mathrm{i}^{d/2-1}\gamma^0\gamma^1 \dots \gamma^{d-1}. \quad (3.4)$$

Model (2): The U(1) is treated as a U(1) gauge group for the dynamical gauge anomaly. The so-called path integral or partition function $\mathbf{Z}$ of this Dirac fermion coupled to a dynamicsl U(1) gauge

field theory is defined as a functional integral (here in Minkowski signature):

$$\begin{aligned}\mathbf{Z} &:= \int [D\bar{\Psi}][D\Psi][DA] \exp\big(+\mathrm{i} S_{M,\text{Dirac-U(1) gauge theory}}\big)\\ &\equiv \int [D\bar{\Psi}][D\Psi][DA] \exp\big(+\mathrm{i}\int_{M^d} \mathrm{d}^d x(\bar{\Psi}(\mathrm{i}\not{D}_A)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu})\big). \end{aligned} \tag{3.5}$$

Here the dynamical 1-form U(1) gauge field A is integrated over in the path integral measure $\int[DA]$ as a dynamical gauge variable. For the quantum electrodynamics (QED) as a Dirac-U(1) gauge theory, it is commonly defined as $\not{D}_A := \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu + \mathrm{i} e A_\mu)$ where e is the electric charge constant.

For the spacetime index $\mu = 0, 1, \dots, d-1$,
the left-moving current $J^{\mu,\mathrm{L}}$ is defined as:

$$J^{\mu,\mathrm{L}} := \bar{\Psi}\gamma^\mu(\frac{1-\gamma^{\text{Chiral}}}{2})\Psi. \tag{3.6}$$

The right-moving current $J^{\mu,\mathrm{R}}$ is defined as:

$$J^{\mu,\mathrm{R}} := \bar{\Psi}\gamma^\mu(\frac{1+\gamma^{\text{Chiral}}}{2})\Psi. \tag{3.7}$$

The vector current $J^\mu \equiv J^{\mu,\mathrm{V}}$ is defined as:

$$J^{\mu,\mathrm{V}} := J^{\mu,\mathrm{L}} + J^{\mu,\mathrm{R}} \equiv \bar{\Psi}\gamma^\mu\Psi. \tag{3.8}$$

The axial chiral current $J^\mu \equiv J^{\mu,\mathrm{A}} \equiv J^\mu \equiv J^{\mu,\text{Chiral}}$ is defined as:

$$J^{\mu,\mathrm{A}} := J^{\mu,\mathrm{L}} - J^{\mu,\mathrm{R}} \equiv \bar{\Psi}\gamma^\mu\gamma^{\text{Chiral}}\Psi. \tag{3.9}$$

We also define the left and right-handed Weyl fermions, Ψ_L and Ψ_R, projected from the Dirac fermion via:

$$\Psi_\mathrm{L} := (\frac{1-\gamma^{\text{Chiral}}}{2})\Psi, \tag{3.10}$$

$$\Psi_\mathrm{R} := (\frac{1+\gamma^{\text{Chiral}}}{2})\Psi. \tag{3.11}$$

In the classical theory (without doing the path integral), the classical Dirac theory has both the continuous vector symmetry $\mathrm{U}(1)_\mathrm{V}$ and the continuous axial symmetry (or the so-called chiral symmetry) $\mathrm{U}(1)_\mathrm{A}$, given by the following symmetry transformation:

$$\mathrm{U}(1)_\mathrm{V}: \quad \begin{aligned}&\Psi \to \exp(\mathrm{i}\alpha_\mathrm{V})\Psi\\ &\Psi_\mathrm{L} \to \exp(\mathrm{i}\alpha_\mathrm{V})\Psi_\mathrm{L}\\ &\Psi_\mathrm{R} \to \exp(\mathrm{i}\alpha_\mathrm{V})\Psi_\mathrm{R}.\end{aligned} \tag{3.12}$$

$$\mathrm{U}(1)_\mathrm{A}: \quad \begin{aligned}&\Psi \to \exp(\mathrm{i}\alpha_\mathrm{A}\gamma^{\text{Chiral}})\Psi\\ &\Psi_\mathrm{L} \to \exp(\mathrm{i}\alpha_\mathrm{A})\Psi_\mathrm{L}\\ &\Psi_\mathrm{R} \to \exp(-\mathrm{i}\alpha_\mathrm{A})\Psi_\mathrm{R}.\end{aligned} \tag{3.13}$$

Under the Noether theorem, the corresponding continuous currents are $J^{\mu,\mathrm{V}}$ and $J^{\mu,\mathrm{A}}$ respect to the $\mathrm{U}(1)_\mathrm{V}$ and $\mathrm{U}(1)_\mathrm{A}$ symmetry respectively. In a classical theory, both $\mathrm{U}(1)_\mathrm{V}$ and $\mathrm{U}(1)_\mathrm{A}$ symmetries are global symmetries.

However, in the quantum theory, we need do the path integral to get the partition function $\mathbf{Z}[A]$ for a quantum theory in eqn. (3.1). Now under the continuous axial (or chiral) symmetry transformation labeled by a U(1) parameter $\alpha_{\rm A} \in [0, 2\pi)$, the partition function $\mathbf{Z}[A]$ shifts to

$$\mathbf{Z}[A] \to \int [D\bar{\Psi}][D\Psi] \exp\big(+ \mathrm{i} \int_{M^d} \mathrm{d}^d x (\bar{\Psi}(\mathrm{i} \not{D}_A)\Psi + \alpha_{\rm A}\Big(\partial_\mu J^{\mu,\text{Chiral}} + \frac{2g^{d/2}}{(d/2)!(4\pi)^{d/2}} \epsilon^{\mu_1\mu_2\dots\mu_d} F_{\mu_1\mu_2}\dots F_{\mu_{d-1}\mu_d}\Big))\big), \quad (3.14)$$

(In terms of quantum electrodynamics notation, people set the $g = -e$.) This means the axial (chiral) current is *not* conserved $\partial_\mu J^{\mu,\text{Chiral}} \neq 0$: If the classical gauge field A has a nontrivial background for $F \wedge \dots F$ term in Model (1) where $F = \mathrm{d}A$. The above calculation can be done based on the Fujikawa's path integral method [59].

The non-conservation of the axial (chiral) current has the form:

$$\partial_\mu J^{\mu,\text{Chiral}} = -\frac{2g^{d/2}}{(d/2)!(4\pi)^{d/2}} \epsilon^{\mu_1\mu_2\dots\mu_d} F_{\mu_1\mu_2}\dots F_{\mu_{d-1}\mu_d}. \quad (3.15)$$

In the differential form in terms of the top form paired with the fundamental class of the spacetime manifold, we can rewrite the above formula eqn. (3.15) as:

$$(\mathrm{d} \star J^{\text{Chiral}}) \propto -g^{d/2}(F \wedge \dots \wedge F). \quad (3.16)$$

The above formula is for the 't Hooft anomaly associated with the probed background Abelian gauge fields (here U(1)).

If we instead consider the background non-Abelian gauge fields (like SU(n)), then the $(F\wedge\dots\wedge F)$ with $F = \mathrm{d}A$ is replaced to a non-abelian field strength $\mathbf{F} = \mathrm{d}\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$; while $(F \wedge \dots \wedge F)$ is replaced by $\mathrm{Tr}\left(\mathbf{F}^{d/2}\right)$:

$$\mathrm{d}\omega_{d-1} = \mathrm{Tr}\left(\mathbf{F}^{d/2}\right). \quad (3.17)$$

The ω_{d-1} is the Chern-Simons $(d-1)$-form [60] as the secondary characteristic classes:

$$\begin{aligned} \omega_1 &\propto \mathrm{Tr}[\mathbf{A}] \\ \omega_3 &\propto \mathrm{Tr}\left[\mathbf{F}\wedge\mathbf{A} - \frac{1}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}\right] \\ \omega_5 &\propto \mathrm{Tr}\left[\mathbf{F}\wedge\mathbf{F}\wedge\mathbf{A} - \frac{1}{2}\mathbf{F}\wedge\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A} + \frac{1}{10}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}\right] \\ &\dots \end{aligned} \quad (3.18)$$

Other than Fujikawa's path integral method [59], we can also capture the perturbative anomaly via a 1-loop Feynman-Dyson diagram calculation. The vertex term means $g\bar{\Psi}\gamma^\mu A_\mu \Psi := g\bar{\Psi}\not{A}\Psi$

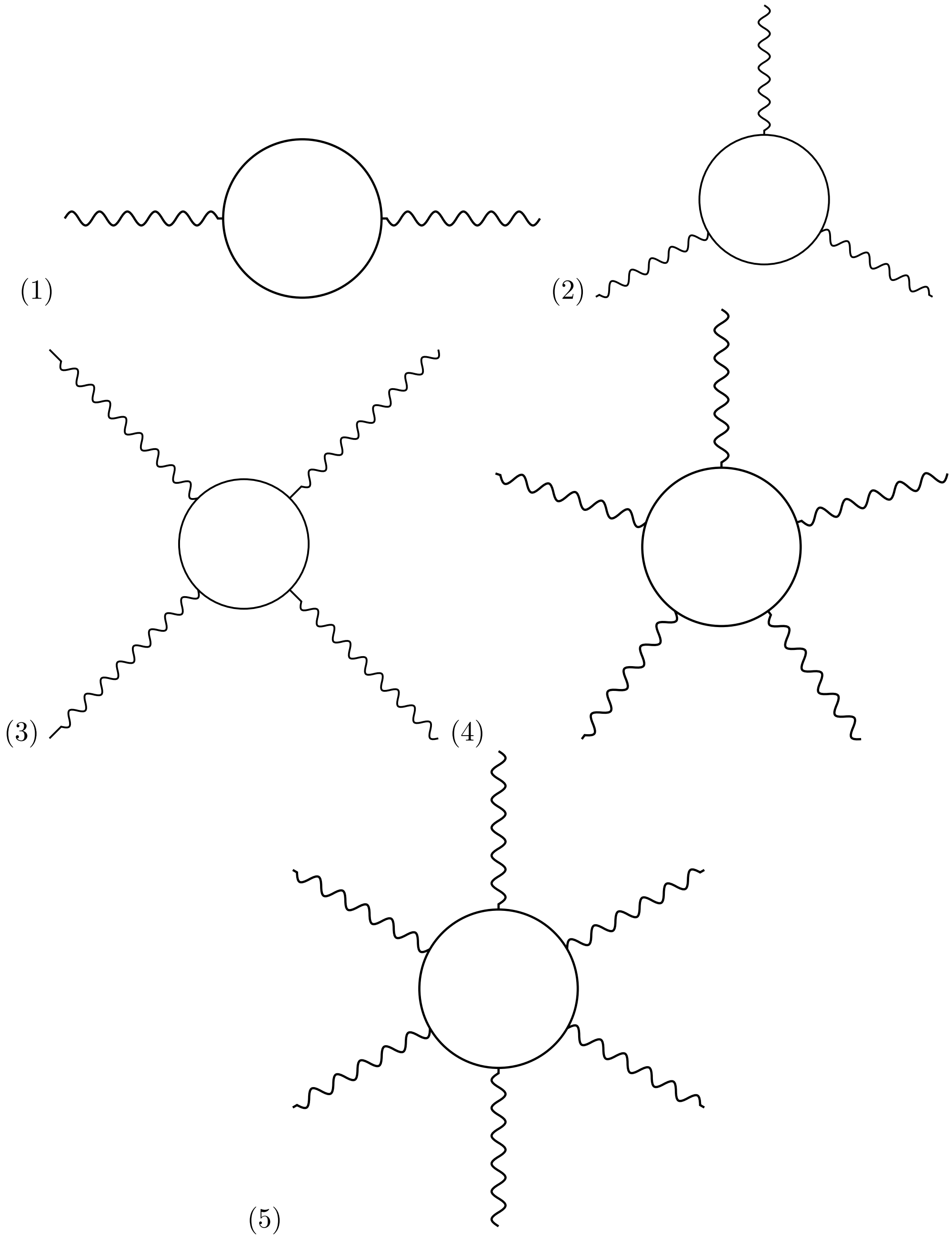


Figure 7: Perturbative anomalies with an integer $\mathbb{Z}$ class in an even d-dimensional spacetime. Here the anomalies are captured by a 1-loop Feynman diagram with a number $(\frac{d}{2}+1)$ amount of vertex terms $g\bar{\Psi}\not{A}\Psi$ and one vertex term among the total $(\frac{d}{2}+1)$ vertex terms can be associated with the J^{Chiral} current. The chiral fermion runs on the solid-line loop (—). The wavy line (∼∼) represents the propagator (Green's function) of 1-form vector boson gauge field. (The gauge field is a probed background gauge field for the 't Hooft anomaly.) In the subfigures, we show 1-loop Feynman diagrams for (1) 2d anomaly, (2) 4d anomaly, (3) 6d anomaly, (4) 8d anomaly, and (5) 10d anomaly, etc., of chiral fermions coupled to U(1) background probed gauge fields. A physical explanation of $\mathbb{Z}$ class is given in Sec. 3.1.2 for 2d bosonic anomaly and Sec. 3.1.3 for 2d fermionic anomaly. Similarly, the analysis can be generalized to any even dd by writing down a one-higher dimensional Chern-Simons theory given by Chern-Simons $(d+1)$d form.

How do we obtain an integer $\mathbb{Z}$ class for perturbative anomalies in an even d-dimensional spacetime? Say from the formulas of eqn. (3.15) and eqn. (3.16)? The answer is that we can consider the modified axial symmetry transformation $\mathrm{U}(1)_{\mathrm{A},k}$ labeled by a integer charge $k \in \mathbb{Z}$ class, such that the Ψ_{L} and Ψ_{R} transformed differently,

$$\begin{aligned} \mathrm{U}(1)_{\mathrm{A},k}: \quad & \Psi_{\mathrm{L}} \to \exp(\mathrm{i}\alpha_{\mathrm{A},k})\Psi_{\mathrm{L}}, \\ & \Psi_{\mathrm{R}} \to \exp(-\mathrm{i}k\alpha_{\mathrm{A},k})\Psi_{\mathrm{R}}. \\ & k \in \mathbb{Z}. \end{aligned} \tag{3.19}$$

The chiral symmetry transformation is labeled by a U(1) parameter $\alpha_{\mathrm{A},k} \in [0, 2\pi)$. Below we give a physical explanation of a $\mathbb{Z}$ class in Sec. 3.1.2 for 2d bosonic anomaly, and also of a $\mathbb{Z}$ class Sec. 3.1.3 for 2d fermionic anomaly. Below our derivation is in a similar spirit of 2d anomaly and 3d Chern-Simons theory [42, 61], but ours is generalizable to arbitary dimensions analogs to [42].

In general, we suggest that the for generic even dd U(1) bosonic or fermionic anomalies (on non-spin or spin manifolds respectively) can be captured by a partition function depending on the probed U(1) background gauge field A:

$$\mathbf{Z}[A] = \exp[\mathrm{i}\frac{c \cdot k}{(2\pi)^{d/2}} \int A \wedge (F)^{d/2}], \quad k \in \mathbb{Z} \tag{3.20}$$

where the precise normalization c depends on the dimensions d, and non-spin or spin manifolds, with the integer $k \in \mathbb{Z}$ class.

3.1.2 2d anomaly and 3d bosonic-U(1) SPTs: Integer $\mathbb{Z}$ class $\in \mathrm{TP}_3(\mathrm{SO} \times \mathrm{U}(1)) = \mathbb{Z}^2$

Below we explicitly derive the analogous eqn. (3.20) for $d = 2$ bosonic anomaly (on non-spin manifolds). The physics idea is that we write down the internal field theory with dynamical gauge fields a coupled to background non-dynamical gauge fields A, and integrate out those internal degrees of freedom to get a response theory depending on A.

The symmetric bilinear form K matrix Chern-Simons theory with a $\mathrm{U}(1)^2$ gauge group of internal dynamical a gauge field, and the charge q vector coupling to the background U(1) gauge fields A are the following:

$$K := K_{IJ} = \begin{pmatrix} 0 & 1 \\ 1 & 2k \end{pmatrix}_{IJ},$$

and $q^T = (1, 1)$. In other words, the path integral, written by dynamical gauge fields a and background fields A, is

$$\mathbf{Z}[A] = \int [Da] \exp[\mathrm{i}\big(\frac{1}{4\pi}\begin{pmatrix} 0 & 1 \\ 1 & 2k \end{pmatrix}_{IJ} \int a_I \wedge \mathrm{d}a_J + \frac{1}{2\pi} q_I^T \int A \wedge \mathrm{d}a_I|_{q^T=(1,1)}\big)], \tag{3.21}$$

Under $\mathrm{GL}(2,\mathbb{Z})$ or $\mathrm{SL}(2,\mathbb{Z})$ redefinition of gauge fields, the K can be changed to $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $q^T = (1, k)$. In other words, the path integral is

$$\mathbf{Z}[A] = \int [Da] \exp[\mathrm{i}\big(\frac{1}{4\pi}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{IJ} \int a_I \wedge \mathrm{d}a_J + \frac{1}{2\pi} q_I^T \int A \wedge \mathrm{d}a_I|_{q^T=(1,k)}\big)], \tag{3.22}$$

If we integrate out the dynamical internal gauge field a ("emergent" from the gapped matter field) of SPTs, we obtain the partition function of probed background field

$$\mathbf{Z}[A] = \exp[\mathrm{i}\,\frac{2k}{4\pi}\int A \wedge \mathrm{d}A], \quad k \in \mathbb{Z} \tag{3.23}$$

This Chern-Simons field theory characterizes the low energy physics of a quantum Hall state and its response function. So the effective bulk quantized Hall conductance is labeled by $2k$ in $2\mathbb{Z}$, as

$$\sigma_{xy} = \frac{qK^{-1}q}{2\pi}(\frac{e^2}{\hbar}) = \frac{2k}{2\pi}(\frac{e^2}{\hbar}) = 2k(\frac{e^2}{h}).$$

The boundary theory has a $\mathbb{Z}$ class of perturbative Adler-Bell-Jackiw type of U(1)-axial-background gauge anomaly. (Due to the L and R chiral fermion carrying imbalanced U(1) charges, in $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $q = (1, k)$.)

The above physics derivation coincides with the mathematical cobordism group calculation, matching one of the integer $\mathbb{Z}$ class $\in \mathrm{TP}_3(\mathrm{SO} \times \mathrm{U}(1)) = \mathbb{Z}^2$ shown later in our Theorem 13. The reason we require the (co)bordism group of $\mathrm{SO} \times \mathrm{U}(1)$ is due to that the bosonic system has a continuous spacetime $\mathrm{SO}(d)$ symmetry (in the dd Euclidean signature), while the boson has an internal U(1) symmetry.

Similarly, the above analysis can be generalized to any even dd by writing down a one-higher dimensional bosonic (on manifolds with $\mathrm{SO}(d+1)$ or non-spin structures) Chern-Simons theory given by a certain Chern-Simons $(d+1)$d form.

3.1.3 2d anomaly and 3d fermionic-U(1) SPTs: Integer $\mathbb{Z}$ class $\in \mathrm{TP}_3(\frac{\mathrm{Spin}\times\mathrm{U}(1)}{\mathbb{Z}_2}) = \mathbb{Z}^2$

Similar to Sec. 3.1.2, for a fermionic theory, we should have a SPT invariant of U(1) background gauge field,

$$\mathbf{Z}[A] = \exp[\mathrm{i}\,\frac{k}{4\pi}\int A \wedge \mathrm{d}A], \quad k \in \mathbb{Z}. \tag{3.24}$$

This may be obtained from integrating out the fermionic SPTs $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $q^T = (1, -k+1)$ or simply $q^T = (0, -k)$. In other words, the path integral is

$$\mathbf{Z}[A] = \int [Da] \exp[\mathrm{i}\,(\frac{1}{4\pi}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{IJ}\int a_I \wedge \mathrm{d}a_J + \frac{1}{2\pi} q_I^T \int A \wedge \mathrm{d}a_I|_{q^T=(1,-k+1)})], \tag{3.25}$$

This Chern-Simons field theory characterizes the low energy physics of another fermionic quantum Hall state and its response function. So the effective bulk quantized Hall conductance is labeled by k in $\mathbb{Z}$, as

$$\sigma_{xy} = \frac{qK^{-1}q}{2\pi}(\frac{e^2}{\hbar}) = \frac{k}{2\pi}(\frac{e^2}{\hbar}) = k(\frac{e^2}{h}).$$

The above physics derivation coincides with the mathematical cobordism group calculation, matching one of the integer $\mathbb{Z}$ class $\in \mathrm{TP}_3(\frac{\mathrm{Spin}\times\mathrm{U}(1)}{\mathbb{Z}_2}) = \mathrm{TP}_3(\mathrm{Spin}^c) = \mathbb{Z}^2$ shown later in our Theorem 17.

The reason we require the (co)bordism group of $(\frac{\mathrm{Spin}\times \mathrm{U}(1)}{\mathbb{Z}_2}) \equiv \mathrm{Spin}^c$ is due to that this fermionic system has a continuous spacetime Spin(d) symmetry (in the dd Euclidean signature) under, the extension of SO(d) via $1 \to \mathbb{Z}_2^F \to \mathrm{Spin}(d) \to \mathrm{SO}(d) \to 1$, while the fermion has an internal U(1) $\supset \mathbb{Z}_2^F$ containing the fermion parity symmetry. A common normal subgroup $\mathbb{Z}_2^F$ is mod out due to the fact that rotating a fermion by 2π in the spacetime (i.e., the spin statistics) gives rise to the same fermion parity minus sign for the fermion field $\Psi \to -\Psi$.

Similarly, the above analysis can be generalized to any even dd by writing down a one-higher dimensional fermionic (on manifolds with Spin($d+1$) thus spin structures) Chern-Simons theory given by a certain Chern-Simons $(d+1)$d form.

3.1.4 $\Omega_d^{\mathrm{SO}}(\mathrm{BU}(1))$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO} \wedge \mathrm{BU}(1)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{SO} \wedge \mathrm{BU}(1)_+)_2^{\wedge} = \Omega_{t-s}^{\mathrm{SO}}(\mathrm{BU}(1)). \quad (3.26)$$

The mod 2 cohomology of Thom spectrum MSO is

$$\mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_2) = \mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \oplus \Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \oplus \Sigma^5\mathcal{A}_2 \oplus \cdots . \quad (3.27)$$

$$\cdots \longrightarrow \Sigma^3\mathcal{A}_2 \longrightarrow \Sigma^2\mathcal{A}_2 \longrightarrow \Sigma\mathcal{A}_2 \longrightarrow \mathcal{A}_2 \longrightarrow \mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \quad (3.28)$$

is an $\mathcal{A}_2$-resolution where the differentials d_1 are induced by Sq^1.

We also have

$$\mathrm{H}^*(\mathrm{BU}(1), \mathbb{Z}_2) = \mathbb{Z}_2[c_1] \quad (3.29)$$

where c_1 is the first Chern class of the universal U(1) bundle.

The E_2 page is shown in Figure 8.

Hence we have the following theorem

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BU}(1))$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}$
3	0
4	$\mathbb{Z}^2$
5	$\mathbb{Z}_2$
6	$\mathbb{Z}^2$
7	$\mathbb{Z}_2$

Theorem 12.

The bordism invariant of $\Omega_2^{\mathrm{SO}}(\mathrm{BU}(1))$ is c_1.

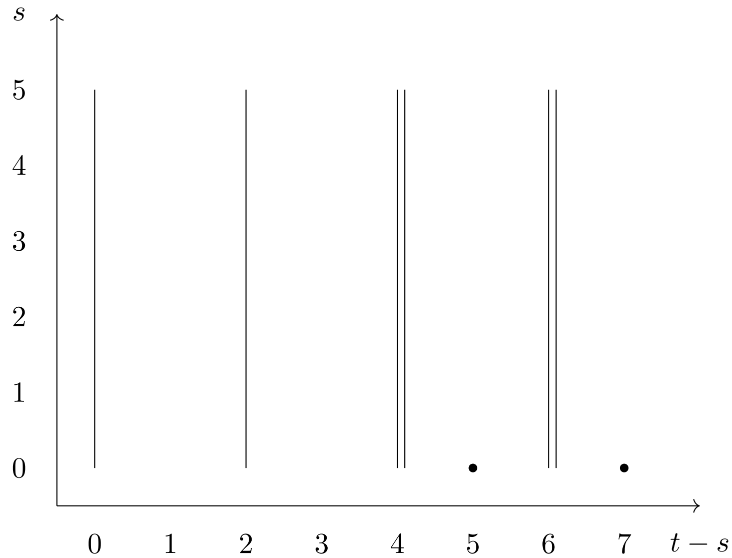


Figure 8: $\Omega_*^{\mathrm{SO}}(\mathrm{BU}(1))$

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{BU}(1))$ are σ, c_1^2.

Here σ is the signature of a 4-manifold.

The bordism invariant of $\Omega_5^{\mathrm{SO}}(\mathrm{BU}(1))$ is w_2w_3.

The bordism invariants of $\Omega_6^{\mathrm{SO}}(\mathrm{BU}(1))$ are $\sigma c_1, c_1^3$.

Here $\sigma c_1 = \sigma(\mathrm{PD}(c_1))$ where $\mathrm{PD}(c_1)$ is the submanifold of the 6-manifold which represents the Poincaé dual of c_1.

The bordism invariant of $\Omega_7^{\mathrm{SO}}(\mathrm{BU}(1))$ is $c_1w_2w_3$.

i	$\mathrm{TP}_i(\mathrm{SO}\times\mathrm{U}(1))$
0	0
1	$\mathbb{Z}$
2	0
3	$\mathbb{Z}^2$
4	0
5	$\mathbb{Z}^2\times\mathbb{Z}_2$
6	0

Table 9: Note that one of the $\mathbb{Z}$ classes in $\mathrm{TP}_3(\mathrm{SO}\times\mathrm{U}(1))=\mathbb{Z}^2$ is given by eqn. (3.22). Similarly, one of the $\mathbb{Z}$ classes in $\mathrm{TP}_{d+1}(\mathrm{SO}\times\mathrm{U}(1))$ for an even d is given by eqn. (3.20).

Theorem 13.

The 1d topological term is the Chern-Simons 1-form $\mathrm{CS}_1^{(\mathrm{U}(1))}$ of the U(1) bundle.

The 3d topological terms are $\frac{1}{3}\mathrm{CS}_3^{(TM)}$ and $\mathrm{CS}_1^{(\mathrm{U}(1))}c_1$ where $\mathrm{CS}_3^{(TM)}$ is the Chern-Simons 3-form

of the tangent bundle.

The 5d topological terms are $c_1 \frac{1}{3}\mathrm{CS}_1^{(TM)}$, $\mathrm{CS}_1^{(\mathrm{U}(1))} c_1^2$ and $w_2 w_3$.

3.1.5 $\Omega_d^{\mathrm{Spin}}(\mathrm{BU}(1))$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin} \wedge \mathrm{BU}(1)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{Spin} \wedge \mathrm{BU}(1)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BU}(1)). \quad (3.30)$$

The mod 2 cohomology of Thom spectrum $M\mathrm{Spin}$ is

$$\mathrm{H}^*(M\mathrm{Spin}, \mathbb{Z}_2) = \mathcal{A}_2 \otimes_{\mathcal{A}_2(1)} \{\mathbb{Z}_2 \oplus M\} \quad (3.31)$$

where M is a graded $\mathcal{A}_2(1)$-module with the degree i homogeneous part $M_i = 0$ for $i < 8$. Here $\mathcal{A}_2(1)$ stands for the subalgebra of $\mathcal{A}_2$ generated by Sq^1 and Sq^2. For $t - s < 8$, we can identify the E_2-page with

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{BU}(1), \mathbb{Z}_2), \mathbb{Z}_2).$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BU}(1), \mathbb{Z}_2)$ is shown in Figure 9.



Figure 9: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BU}(1), \mathbb{Z}_2)$.

The E_2 page is shown in Figure 10.

Hence we have the following theorem

i	$\Omega_i^{\mathrm{Spin}}(\mathrm{BU}(1))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z} \times \mathbb{Z}_2$
3	0
4	$\mathbb{Z}^2$
5	0
6	$\mathbb{Z}^2$
7	0

Theorem 14.

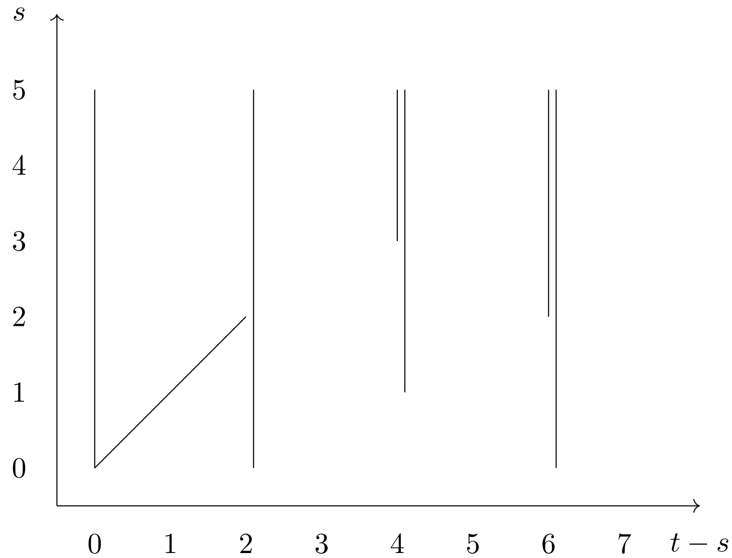


Figure 10: $\Omega_*^{\text{Spin}}(\text{BU}(1))$

The bordism invariant of $\Omega_1^{\text{Spin}}(\text{BU}(1))$ is $\tilde{\eta}$.

Here $\tilde{\eta}$ is the "mod 2 index" of the 1d Dirac operator (#zero eigenvalues mod 2, no contribution from spectral asymmetry).

The bordism invariants of $\Omega_2^{\text{Spin}}(\text{BU}(1))$ are c_1 and Arf (the Arf invariant).

The bordism invariants of $\Omega_4^{\text{Spin}}(\text{BU}(1))$ are $\frac{\sigma}{16}$ and $\frac{c_1^2}{2}$.

Here c_1^2 is divided by 2 since $c_1^2 = \text{Sq}^2 c_1 = (w_2(TM) + w_1(TM)^2)c_1 = 0 \mod 2$ on Spin 4-manifolds.

The bordism invariants of $\Omega_6^{\text{Spin}}(\text{BU}(1))$ are c_1^3 and $\frac{c_1(\sigma - F\cdot F)}{8}$.

Here $\frac{c_1(\sigma-F\cdot F)}{8}$ is defined to be $\frac{1}{8}(\sigma(\text{PD}(c_1)) - F\cdot F)$ where $\text{PD}(c_1)$ is the submanifold of a Spin 6-manifold which represents the Poincaré dual of c_1, $\sigma(\text{PD}(c_1))$ is the signature of $\text{PD}(c_1)$, and F is a characteristic surface of $\text{PD}(c_1)$. By Rokhlin's theorem, $\sigma(\text{PD}(c_1)) - F\cdot F$ is a multiple of 8 and $\frac{1}{8}(\sigma(\text{PD}(c_1)) - F\cdot F) = \text{Arf}(\text{PD}(c_1), F) \mod 2$. See [62]'s Lecture 10 for more details.

i	$\text{TP}_i(\text{Spin}\times \text{U}(1))$
0	0
1	$\mathbb{Z}\times\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}^2$
4	0
5	$\mathbb{Z}^2$
6	0

Theorem 15.

The 1d topological terms are $\mathrm{CS}_1^{(\mathrm{U}(1))}$ and $\tilde{\eta}$.

The 2d topological term is Arf.

The 3d topological terms are $\frac{1}{48}\mathrm{CS}_3^{(TM)}$ and $\frac{1}{2}\mathrm{CS}_1^{(\mathrm{U}(1))}c_1$.

The 5d topological terms are $\mathrm{CS}_1^{(\mathrm{U}(1))}c_1^2$ and $\mu(\mathrm{PD}(c_1))$.

Here $\mu(\mathrm{PD}(c_1))$ is the Rokhlin invariant (see [62]'s Lecture 11) of $\mathrm{PD}(c_1)$ where $\mathrm{PD}(c_1)$ is the submanifold of a Spin 5-manifold which represents the Poincaré dual of c_1

3.1.6 $\Omega_d^{(\frac{\mathrm{Spin}\times\mathrm{U}(1)}{\mathbb{Z}_2})} = \Omega_d^{\mathrm{Spin}^c}$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(MT\mathrm{Spin}^c,\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \pi_{t-s}(MT\mathrm{Spin}^c)_2^\wedge = \Omega_{t-s}^{\mathrm{Spin}^c}. \tag{3.32}$$

By 1801.07530, we have $MT\mathrm{Spin}^c = M\mathrm{Spin}\wedge\Sigma^{-2}M\mathrm{U}(1)$.

The mod 2 cohomology of Thom spectrum $M\mathrm{Spin}$ is

$$\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_2) = \mathcal{A}_2\otimes_{\mathcal{A}_2(1)}\{\mathbb{Z}_2\oplus M\} \tag{3.33}$$

where M is a graded $\mathcal{A}_2(1)$-module with the degree i homogeneous part $M_i = 0$ for $i<8$. Here $\mathcal{A}_2(1)$ stands for the subalgebra of $\mathcal{A}_2$ generated by Sq^1 and Sq^2. For $t-s<8$, we can identify the E_2-page with

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+2}(M\mathrm{U}(1),\mathbb{Z}_2),\mathbb{Z}_2).$$

By Thom's isomorphism, $\mathrm{H}^{*+2}(M\mathrm{U}(1),\mathbb{Z}_2) = \mathbb{Z}_2[c_1]U$ where U is the Thom class of the universal U(1) bundle and c_1 is the first Chern class of the universal U(1) bundle.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(M\mathrm{U}(1),\mathbb{Z}_2)$ is shown in Figure 11.

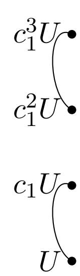


Figure 11: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+2}(M\mathrm{U}(1),\mathbb{Z}_2)$.

The E_2 page is shown in Figure 12.

Hence we have the following theorem

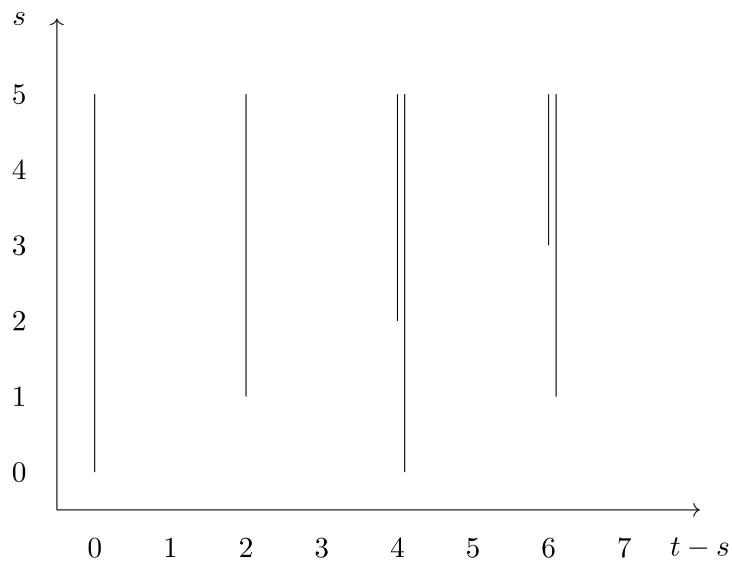


Figure 12: $\Omega_*^{\mathrm{Spin}^c}$

i	$\Omega_i^{\mathrm{Spin}^c}$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}$
3	0
4	$\mathbb{Z}^2$
5	0
6	$\mathbb{Z}^2$
7	0

Theorem 16.

The bordism invariants of $\Omega_2^{\mathrm{Spin}^c}$ is $\frac{c_1}{2}$.

Here c_1 is divided by 2 since $c_1 \mod 2 = w_2(TM)$ while $w_2(TM) = 0$ on Spin^c 2-manifolds.

The bordism invariants of $\Omega_4^{\mathrm{Spin}^c}$ are c_1^2 and $\frac{\sigma - F\cdot F}{8}$.

Here F is a characteristic surface of the Spin^c 4-manifold M. By Rokhlin's theorem, $\sigma - F \cdot F$ is a multiple of 8 and $\frac{1}{8}(\sigma - F \cdot F) = \mathrm{Arf}(M, F) \mod 2$. See [62]'s Lecture 10 for more details.

The bordism invariants of $\Omega_6^{\mathrm{Spin}^c}$ are $\frac{c_1^3}{2}$ and $c_1\frac{\sigma}{16}$.

Here $c_1\frac{\sigma}{16} = \frac{\sigma}{16}(\mathrm{PD}(c_1))$ where $\mathrm{PD}(c_1)$ is the submanifold of the Spin^c 6-manifold which represents the Poincaré dual of c_1. Note that $\mathrm{PD}(c_1)$ is Spin.

Theorem 17.

The 1d topological term is $\frac{1}{2}\mathrm{CS}_1^{(\mathrm{U}(1))}$.

i	$\mathrm{TP}_i(\mathrm{Spin}^c)$
0	0
1	$\mathbb{Z}$
2	0
3	$\mathbb{Z}^2$
4	0
5	$\mathbb{Z}^2$
6	0

Table 10: Note that one of the $\mathbb{Z}$ classes in $\mathrm{TP}_3(\mathrm{Spin}^c) = \mathbb{Z}^2$ is given by eqn. (3.24). Similarly, one of the $\mathbb{Z}$ classes in $\mathrm{TP}_{d+1}(\mathrm{Spin}^c)$ for an even d is given by eqn. (3.20).

The 3d topological terms are $\mathrm{CS}_1^{(\mathrm{U}(1))} c_1$ and μ.

Here μ is the Rokhlin invariant (see [62]'s Lecture 11) of the Spin^c 3-manifold.

The 5d topological terms are $\frac{1}{2}\mathrm{CS}_1^{(\mathrm{U}(1))} c_1^2$ and $c_1 \frac{1}{48}\mathrm{CS}_1^{(TM)}$.

3.2 Non-perturbative global anomalies: Witten's SU(2) Anomaly and A New SU(2) Anomaly in 4d and 5d

We now provide another warm-up example, a $(d+1)$-th cobordism group calculation associated with dd *non-perturbative global* anomalies for the SU(2) anomaly of Witten [23] and the new SU(2) anomaly [24].

Here these SU(2) anomalies will be interpreted as the 't Hooft anomalies of the internal SU(2) global symmetries in the QFT, whose fermion multiplets are only in half-integer isospin (say, 1/2, 3/2, 5/2, . . .)-representation of SU(2); while whose bosons are only in an integer isospin (say, 0, 1, 2, . . .)-representation of SU(2).

Similar to the $\mathrm{Spin}^c \equiv (\frac{\mathrm{Spin}\times\mathrm{U}(1)}{\mathbb{Z}_2})$ of Sec. 3.1.3, in the following subsections, we will study the (co)bordism group of $(\frac{\mathrm{Spin}\times\mathrm{SU}(2)}{\mathbb{Z}_2})$.

Here the fermionic system has a continuous spacetime $\mathrm{Spin}(d)$ symmetry (in the dd Euclidean signature) under, the extension of $\mathrm{SO}(d)$ via $1 \to \mathbb{Z}_2^F \to \mathrm{Spin}(d) \to \mathrm{SO}(d) \to 1$, while the fermion has an internal $\mathrm{SU}(2) \supset \mathbb{Z}_2^F$ containing the fermion parity symmetry at SU(2)'s center. A common normal subgroup $\mathbb{Z}_2^F$ is mod out due to the fact that rotating a fermion by 2π in the spacetime (i.e., the spin statistics) gives rise to the same fermion parity minus sign for the fermion field $\Psi \to -\Psi$. Thus we need to study the $(\frac{\mathrm{Spin}\times\mathrm{SU}(2)}{\mathbb{Z}_2})$-structure.

We will see that $\mathrm{TP}_5(\frac{\mathrm{Spin}\times\mathrm{SU}(2)}{\mathbb{Z}_2^F}) = (\mathbb{Z}_2)^2$. These 5d bordism invariants generates $(\mathbb{Z}_2)^2$, they correspond to the old SU(2) [23] and the new SU(2) anomalies [24] in 4d, shown in Table 3.2.1.

We will see that $\mathrm{TP}_6(\frac{\mathrm{Spin}\times\mathrm{SU}(2)}{\mathbb{Z}_2^F}) = (\mathbb{Z}_2)^2$. These 6d bordism invariants generates $(\mathbb{Z}_2)^2$, they correspond to the old SU(2) and the new SU(2) anomalies in 5d [24], shown in Table 3.2.1.

3.2.1 $\Omega_d^{\frac{\mathrm{Spin}\times\mathrm{SU}(2)}{\mathbb{Z}_2^F}} = \Omega_d^{\frac{\mathrm{Spin}\times\mathrm{Spin}(3)}{\mathbb{Z}_2^F}}$

Let $H = \frac{\mathrm{Spin}\times\mathrm{Spin}(3)}{\mathbb{Z}_2^F}$, we have a homotopy pullback square

$$\begin{CD} \mathrm{B}H @>>> \mathrm{BSO}(3) \\ @VVV @VV{w_2'}V \\ \mathrm{BSO} @>{w_2(TM)}>> \mathrm{B}^2\mathbb{Z}_2 \end{CD} \tag{3.34}$$

There is a homotopy equivalence $f : \mathrm{BSO}\times\mathrm{BSO}(3) \xrightarrow{\sim} \mathrm{BSO}\times\mathrm{BSO}(3)$ by $(V,W)\mapsto(V-W+3,W)$. Note that $f^*(w_2) = w_2(V-W) = w_2(V)+w_1(V)w_1(W)+w_2(W) = w_2(TM)+w_2'$. Then we have the following homotopy pullback

$$\begin{CD} \mathrm{B}H @>{\sim}>> \mathrm{BSpin}\times\mathrm{BSO}(3) \\ @VVV @VVV \\ \mathrm{BSO}\times\mathrm{BSO}(3) @>{f}>> \mathrm{BSO}\times\mathrm{BSO}(3) @>{w_2+0}>> \mathrm{B}^2\mathbb{Z}_2 \\ @V{(V,W)\mapsto V}VV \\ \mathrm{BSO} \end{CD} \tag{3.35}$$

with $\mathrm{BSO}\times\mathrm{BSO}(3) \xrightarrow{w_2(TM)+w_2'} \mathrm{B}^2\mathbb{Z}_2$ and $\mathrm{BSO}\times\mathrm{BSO}(3) \xrightarrow{(V,W)\mapsto V+W-3} \mathrm{BSO}$.

This implies that $\mathrm{B}H \sim \mathrm{BSpin}\times\mathrm{BSO}(3)$.

$MTH = \mathrm{Thom}(\mathrm{B}H;-V)$, where V is the induced virtual bundle (of dimension 0) by the map $\mathrm{B}H\to\mathrm{BO}$.

We can identify $\mathrm{B}H\to\mathrm{BO}$ with $\mathrm{BSpin}\times\mathrm{BSO}(3) \xrightarrow{V-V_3+3} \mathrm{BSO}\hookrightarrow\mathrm{BO}$.

The spectrum MTH is homotopy equivalent to $\mathrm{Thom}(\mathrm{BSpin}\times\mathrm{BSO}(3);-(V-V_3+3))$, which is $M\mathrm{Spin}\wedge\Sigma^{-3}M\mathrm{SO}(3)$.

For $t-s<8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+3}(M\mathrm{SO}(3),\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\frac{\mathrm{Spin}\times\mathrm{Spin}(3)}{\mathbb{Z}_2^F}}.$$

By Thom's isomorphism, $\mathrm{H}^{*+3}(M\mathrm{SO}(3),\mathbb{Z}_2) = \mathbb{Z}_2[w_2',w_3']U$ where w_i' is the Stiefel-Whitney class of the universal SO(3) bundle and U is the Thom class of the universal SO(3) bundle.

Since $w_1(TM) = w_1' = 0$, and $w_2(TM) = w_2'$ by the gauge bundle constraint, we have $w_3(TM) = w_3'$. Below we use w_i to denote both $w_i(TM)$ and w_i' for $i\le 3$.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+3}(M\mathrm{SO}(3),\mathbb{Z}_2)$ and the E_2 page are shown in Figure 13, 14.

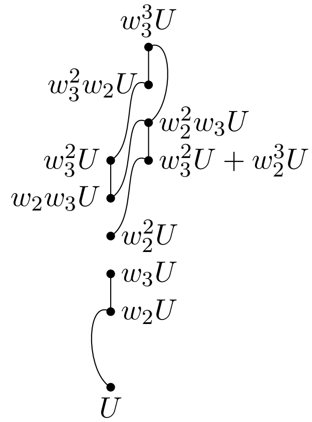


Figure 13: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+3}(M\mathrm{SO}(3), \mathbb{Z}_2)$

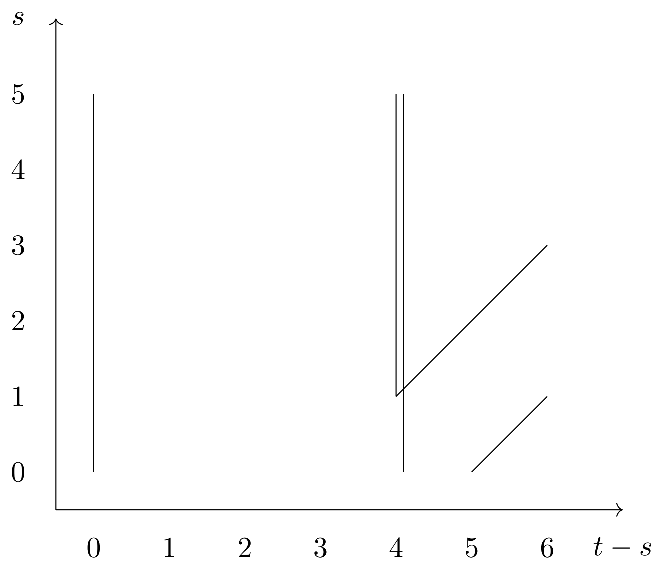


Figure 14: $\Omega_*^{\frac{\mathrm{Spin}\times\mathrm{Spin}(3)}{\mathbb{Z}_2^F}}$

4 Higher Group Cobordisms and Non-trivial Fibrations

In this section, we use the Serre spectral sequence method explored in appendix of [11] and the Adams spectral sequence method to derive the 5d topological terms for the higher group cobordism $\Omega_5^{\mathbb{G}}$ with non-trivial fibration where $\mathbb{G}$ is defined as follows.

If G_a is a group, G_b is an abelian group, then it is well-known that $\mathrm{B}G_b$ is a group. Consider the group extension

$$1 \to \mathrm{B}G_b \to \mathbb{G} \to G_a \to 1, \tag{4.1}$$

Bordism group		
d	$\Omega_d^{\frac{\text{Spin}\times\text{SU}(2)}{\mathbb{Z}_2^F}}$	bordism invariants
0	$\mathbb{Z}$	
1	0	
2	0	
3	0	
4	$\mathbb{Z}^2$	(σ, N_0)
5	$\mathbb{Z}_2^2$	$(w_2w_3,\ \text{Arf}\cup w_3)$
6	$\mathbb{Z}_2^2$	$(w_2w_3\cup\tilde{\eta}, \frac{\sigma(\text{PD}(w_2))}{16} \mod 2))$

Table 11: Bordism group. Here σ is the signature of 4-manifolds, $\text{PD}(w_2)$ is the submanifold of a 6-manifold which represents the Poincaré dual of w_2. Note that $\text{PD}(w_2)$ is Spin. The N_0 is the number of the zero modes of the Dirac operator in 4d. It is defined in [25]. On oriented 4-manifolds, $N_0 = N_0' = N_+ - N_-$ where N_0' is also defined in [25], and $N_\pm$ are the numbers of zero modes of the Dirac operator with given chirality.

Cobordism group		
d	$\text{TP}_d(\frac{\text{Spin}\times\text{SU}(2)}{\mathbb{Z}_2^F})$	topological terms
0	0	
1	0	
2	0	
3	$\mathbb{Z}^2$	$(\frac{1}{3}\text{CS}_3^{(TM)}, X)$
4	0	
5	$\mathbb{Z}_2^2$	$(w_2w_3,\ \text{Arf}\cup w_3)$
6	$\mathbb{Z}_2^2$	$(w_2w_3\cup\tilde{\eta}, \frac{\sigma(\text{PD}(w_2))}{16} \mod 2)$

Table 12: SPT states in d-dim spacetime. $\text{TP}_5(\frac{\text{Spin}\times\text{SU}(2)}{\mathbb{Z}_2^F}) = (\mathbb{Z}_2)^2$. whose 5d bordism invariants correspond to the old SU(2) [23] and the new SU(2) anomalies [24] in 4d. A related cobordism group in one higher dimension, $\text{TP}_6(\frac{\text{Spin}\times\text{SU}(2)}{\mathbb{Z}_2^F}) = (\mathbb{Z}_2)^2$. whose 6d bordism invariants correspond to the old SU(2) and the new SU(2) anomalies in 5d [24]. Since $\Omega_3^{\text{Spin}\times_{\mathbb{Z}_2}\text{SU}(2)} = 0$, for any $\text{Spin}\times_{\mathbb{Z}_2}\text{SU}(2)$ 3-manifold M^3, M^3 is the boundary of a $\text{Spin}\times_{\mathbb{Z}_2}\text{SU}(2)$ 4-manifold M^4, then we define the 3d topological term X on M^3 to be N_0 of M^4. Note that Dirac operator can be defined for manifolds with boundary. If the bulk manifold has Dirac operator which has a mass m being gapped, then the boundary manifold can have Dirac operator no mass $m = 0$ being gapless.

we have a fibration

$$\begin{array}{ccc} \text{B}^2G_b & \longrightarrow & \text{B}\mathbb{G} \\ & & \downarrow \\ & & \text{B}G_a \end{array} \tag{4.2}$$

which is classified by the Postnikov class $\beta \in \text{H}^3(\text{B}G_a, G_b)$.

4.1 $(\mathrm{B}G_a, \mathrm{B}^2 G_b) : (\mathrm{B}\mathbf{O}, \mathrm{B}^2\mathbb{Z}_2)$

We consider the simplest case: $G_a = \mathrm{O}$ and $G_b = \mathbb{Z}_2$. Note that there is also a group action $\alpha : G_a \to \mathrm{Aut}G_b$ in [11], since $\mathrm{Aut}\mathbb{Z}_2$ is trivial, so α is trivial in this special case.

For the fibration

$$\begin{array}{ccc} \mathrm{B}^2\mathbb{Z}_2 & \longrightarrow & \mathrm{B}\mathbb{G} \\ & & \downarrow \\ & & \mathrm{BO}, \end{array} \tag{4.3}$$

there is a Serre spectral sequence

$$\mathrm{H}^p(\mathrm{BO}, \mathrm{H}^q(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(\mathrm{B}\mathbb{G}, \mathbb{Z}) \tag{4.4}$$

where $\mathrm{H}^p(\mathrm{BO}, \mathrm{H}^q(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}))$ actually should be the α-equivariant cohomology, but since α is trivial, $\mathrm{H}^p(\mathrm{BO}, \mathrm{H}^q(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}))$ is the ordinary cohomology.

Note that $\mathrm{H}^n(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z})$ is computed in Appendix C of [63].

$$\mathrm{H}^n(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 1 \\ 0 & n = 2 \\ \mathbb{Z}_2 & n = 3 \\ 0 & n = 4 \\ \mathbb{Z}_4 & n = 5 \\ \mathbb{Z}_2 & n = 6 \end{cases} \tag{4.5}$$

The E_2 page of the Serre spectral sequence is $\mathrm{H}^p(\mathrm{BO}, \mathrm{H}^q(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}))$. The shape of the relevant piece is shown in Figure 15.

Note that p labels the columns and q labels the rows.

The bottom row is $\mathrm{H}^p(\mathrm{BO}, \mathbb{Z})$.

The universal coefficient theorem (2.15) tells us that $\mathrm{H}^3(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}) = \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}), \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(\pi_2(\mathrm{B}^2\mathbb{Z}_2), \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) = \hat{\mathbb{Z}}_2$, so the $q = 3$ row is $\mathrm{H}^p(\mathrm{BO}, \hat{\mathbb{Z}}_2)$.

It is also known that $\mathrm{H}^5(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}) = \mathrm{H}^4(\mathrm{B}^2\mathbb{Z}_2, \mathbb{R}/\mathbb{Z})$ is the group of quadratic functions $q : \mathbb{Z}_2 \to \mathbb{R}/\mathbb{Z}$ [65]. The isomorphism is discussed in detail in [66].

The first possibly non-zero differential is on the E_3 page:

$$\mathrm{H}^0(\mathrm{BO}, \mathrm{H}^5(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z})) \to \mathrm{H}^3(\mathrm{BO}, \hat{\mathbb{Z}}_2). \tag{4.6}$$

Following the appendix of [11], this map sends a quadratic form $q : \mathbb{Z}_2 \to \mathbb{R}/\mathbb{Z}$ to $\langle \beta, - \rangle_q$, where the bracket denotes the bilinear pairing $\langle x, y \rangle_q = q(x + y) - q(x) - q(y)$.

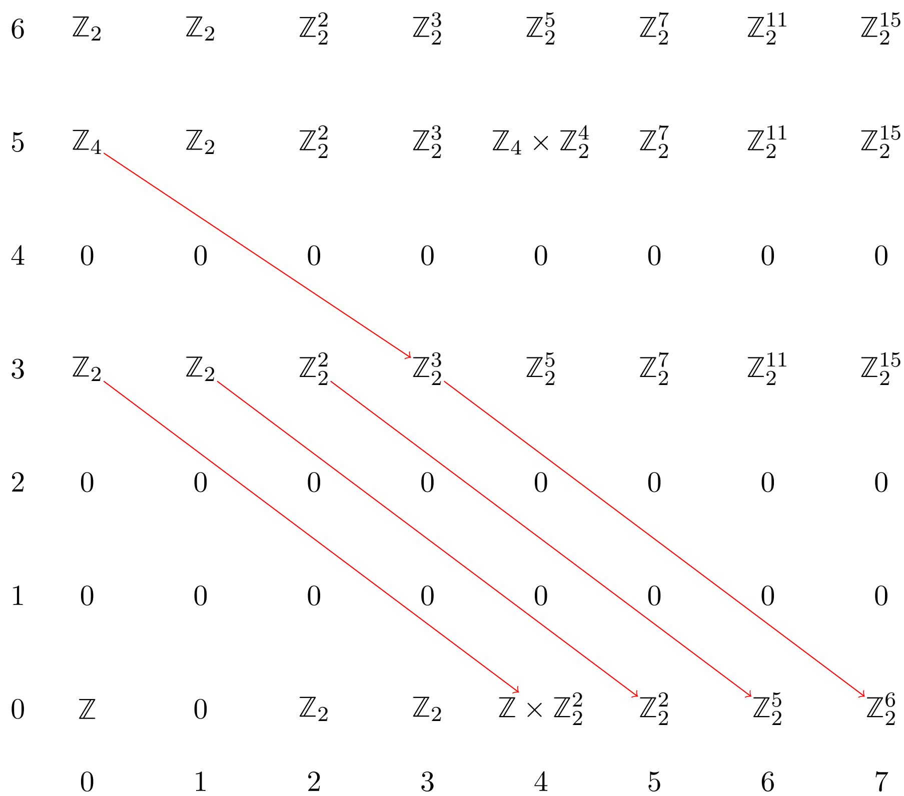


Figure 15: Serre spectral sequence for $(\mathrm{BO}, \mathrm{B}^2\mathbb{Z}_2)$. Here the row $q = 0$ is the result of [64, Theorem 1.6]. The row $q = 3$ is the result that $\mathrm{H}^*(\mathrm{BO}, \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \dots]$ where w_i is the Stiefel-Whitney class of the virtual bundle (of dimension 0) over BO. The rows $q = 0$ and $q = 3$ are related by the universal coefficient theorem (2.20). The row $q = 5$ is resulting from the row $q = 0$ by the universal coefficient theorem (2.20).

The next possibly non-zero differentials are on the E_4 page:

$$\mathrm{H}^j(\mathrm{BO}, \hat{\mathbb{Z}}_2) \to \mathrm{H}^{j+3}(\mathrm{BO}, \mathbb{R}/\mathbb{Z}) \to \mathrm{H}^{j+4}(\mathrm{BO}, \mathbb{Z}). \tag{4.7}$$

The first map is contraction with β. The second map comes from the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^n(\mathrm{BO}, \mathbb{R}) \longrightarrow \mathrm{H}^n(\mathrm{BO}, \mathbb{R}/\mathbb{Z}) \longrightarrow \mathrm{H}^{n+1}(\mathrm{BO}, \mathbb{Z}) \longrightarrow \mathrm{H}^{n+1}(\mathrm{BO}, \mathbb{R}) \longrightarrow \cdots. \tag{4.8}$$

If $\mathrm{H}^n(\mathrm{BO}, \mathbb{R}) = \mathrm{H}^{n+1}(\mathrm{BO}, \mathbb{R}) = 0$, then $\mathrm{H}^n(\mathrm{BO}, \mathbb{R}/\mathbb{Z}) = \mathrm{H}^{n+1}(\mathrm{BO}, \mathbb{Z})$. Since $\mathrm{H}^n(\mathrm{BO}, \mathbb{R}) = \mathrm{H}^n(\mathrm{BO}, \mathbb{Z}) \otimes \mathbb{R}$ and $\mathrm{H}^n(\mathrm{BO}, \mathbb{Z})$ is finite if n is not divisible by 4, $\mathrm{H}^n(\mathrm{BO}, \mathbb{R}) = 0$ if n is not divisible by 4, thus $\mathrm{H}^n(\mathrm{BO}, \mathbb{R}/\mathbb{Z}) = \mathrm{H}^{n+1}(\mathrm{BO}, \mathbb{Z})$ for $n = 1, 2 \mod 4$.

The last relevant possibly non-zero differential is on the E_6 page:

$$\mathrm{H}^0(\mathrm{BO}, \mathrm{H}^5(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z})) \to \mathrm{H}^6(\mathrm{BO}, \mathbb{Z}). \tag{4.9}$$

Following the appendix of [11], this differential is actually zero.

So the only possible differentials in Figure 15 below degree 5 are d_3 from $(0, 5)$ to $(3, 3)$ and d_4 from the third row to the zeroth row.

By the Universal Coefficient Theorem (2.20),

$$\mathrm{H}^n(\mathrm{B}\mathbb{G},\mathbb{Z}_2)=\mathrm{H}^n(\mathrm{B}\mathbb{G},\mathbb{Z})\otimes\mathbb{Z}_2\oplus\mathrm{Tor}(\mathrm{H}^{n+1}(\mathrm{B}\mathbb{G},\mathbb{Z}),\mathbb{Z}_2). \tag{4.10}$$

The Madsen-Tillmann spectrum $MT\mathbb{G}=\mathrm{Thom}(\mathrm{B}\mathbb{G};-V)$ where V is the induced virtual bundle over $\mathrm{B}\mathbb{G}$ (of dimension 0) from $\mathrm{B}\mathbb{G}\to\mathrm{BO}$.

By Thom isomorphism (1.12), $\mathrm{H}^*(MT\mathbb{G},\mathbb{Z}_2)=\mathrm{H}^*(\mathrm{B}\mathbb{G},\mathbb{Z}_2)U$ where U is the Thom class of $-V$ with $\mathrm{Sq}^iU=\bar{w}_iU$ where $\bar{w}_i$ is the Stiefel-Whitney class of $-V$ such that $(1+\bar{w}_1+\bar{w}_2+\cdots)(1+w_1+w_2+\cdots)=1$ where w_i is the Stiefel-Whitney class of V, i.e., $\bar{w}_1=w_1$, $\bar{w}_2=w_2+w_1^2$, etc. Here the U on the right means the cup product with U.

We have the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(MT\mathbb{G},\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\pi_{t-s}(MT\mathbb{G})=\Omega_{t-s}^{\mathbb{G}} \tag{4.11}$$

where $\mathcal{A}_2$ is the mod 2 Steenrod algebra. The last equality is Pontryagin-Thom isomorphism.

The $\mathcal{A}_2$-module structure of $\mathrm{H}^*(MT\mathbb{G},\mathbb{Z}_2)$ below degree 5 is shown in Figure 16 where we intentionally omit terms that don't involve the cohomology classes of $\mathrm{B}^2\mathbb{Z}_2$.

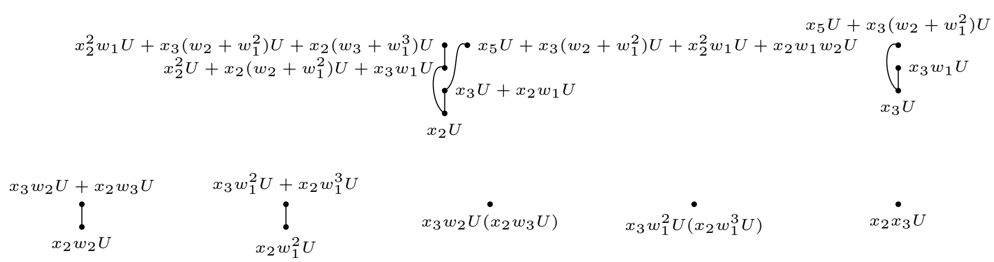


Figure 16: The $\mathcal{A}_2$-module structure of $\mathrm{H}^*(MT\mathbb{G},\mathbb{Z}_2)$ below degree 5. Here x_2 is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$, $x_3=\mathrm{Sq}^1x_2$, and $x_5=\mathrm{Sq}^2x_3$.

Note that the position $(0,3)$ in Figure 15 contributes to both $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ which is generated by x_2 and $\mathrm{H}^3(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ which is generated by x_3, $\mathrm{Sq}^1x_2=x_3$. The position $(2,3)$ corresponds to xw_1^2,xw_2, the position $(3,3)$ corresponds to xw_1^3,xw_1w_2,xw_3 for both $x=x_2$ and $x=x_3$.

Since $\langle\beta,\beta\rangle_q=-2q(\beta)$, $4q(\beta)=q(2\beta)=0$, there are 2 among the 4 choices of $q(\beta)$ such that $q\to\langle\beta,-\rangle_q$ maps to the dual linear function of β, if we identify $\hat{\mathbb{Z}}_2$ with $\mathbb{Z}_2$, then the nonzero element in the image of $q\to\langle\beta,-\rangle_q$ is just β. So $\mathrm{Im}d_3^{(0,5)}$ is spanned by $x\beta$.

The differential $d_4^{(2,3)}:\mathrm{H}^2(\mathrm{BO},\hat{\mathbb{Z}}_2)\to\mathrm{H}^5(\mathrm{BO},\mathbb{R}/\mathbb{Z})=\mathrm{H}^6(\mathrm{BO},\mathbb{Z})$ is defined by

$$d_4^{(2,3)}(\gamma)(v_0,\dots,v_5)=(\gamma(v_0,\dots,v_2))(\beta(v_2,\dots,v_5)).$$

Let $\beta=a_1w_1^3+a_2w_1w_2+a_3w_3$, if we identify $\hat{\mathbb{Z}}_2$ with $\mathbb{Z}_2$, then $d_4^{(2,3)}(\gamma)=\gamma\cup\beta$ which is in $\mathrm{H}^5(\mathrm{BO},\mathbb{Z}_2)$, while $\mathrm{H}^5(\mathrm{BO},\mathbb{R}/\mathbb{Z})=\mathrm{H}^6(\mathrm{BO},\mathbb{Z})$. Let $\gamma=b_1w_1^2+b_2w_2$, then $\gamma\cup\beta=a_1b_1w_1^5+(a_1b_2+a_2b_1)w_1^3w_2+a_2b_2w_1w_2^2+a_3b_1w_1^2w_3+a_3b_2w_2w_3$.

The differential $d_4^{(3,3)}:\mathrm{H}^3(\mathrm{BO},\hat{\mathbb{Z}}_2)\to\mathrm{H}^6(\mathrm{BO},\mathbb{R}/\mathbb{Z})=\mathrm{H}^7(\mathrm{BO},\mathbb{Z})$ is defined by

$$d_4^{(3,3)}(\zeta)(v_0,\dots,v_6)=(\zeta(v_0,\dots,v_3))(\beta(v_3,\dots,v_6)).$$

Let $\beta = a_1w_1^3 + a_2w_1w_2 + a_3w_3$, if we identify $\hat{\mathbb{Z}}_2$ with $\mathbb{Z}_2$, then $d_4^{(3,3)}(\zeta) = \zeta \cup \beta$ which is in $\mathrm{H}^6(\mathrm{BO}, \mathbb{Z}_2)$, while $\mathrm{H}^6(\mathrm{BO}, \mathbb{R}/\mathbb{Z}) = \mathrm{H}^7(\mathrm{BO}, \mathbb{Z})$. Let $\zeta = c_1w_1^3 + c_2w_1w_2 + c_3w_3$, then $\zeta \cup \beta = a_1c_1w_1^6 + a_2c_2w_1^2w_2^2 + a_3c_3w_3^2 + (a_1c_2 + c_1a_2)w_1^4w_2 + (a_1c_3 + c_1a_3)w_1^3w_3 + (a_2c_3 + c_2a_3)w_1w_2w_3$.

Note that by the Universal Coefficient Theorem (2.20),

$$\mathrm{H}^n(\mathrm{BO}, \mathbb{Z}_2) = \mathrm{H}^n(\mathrm{BO}, \mathbb{Z}) \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}(\mathrm{H}^{n+1}(\mathrm{BO}, \mathbb{Z}), \mathbb{Z}_2). \tag{4.12}$$

By [64, Theorem 1.6]:

$$\mathrm{H}^*(\mathrm{BO}, \mathbb{Z}) \otimes \mathbb{Z}_2 = \mathbb{Z}_2[w_2^2, \mathrm{Sq}^1(w_1), \mathrm{Sq}^1(w_2), \mathrm{Sq}^1(w_1w_2), \mathrm{Sq}^1(w_4), \mathrm{Sq}^1(w_1w_4), \mathrm{Sq}^1(w_2w_4), \dots] \tag{4.13}$$

where we list the generators below degree 7. Among the linear combinations of $w_1^5, w_1^3w_2, w_1w_2^2, w_1^2w_3, w_2w_3$, only $w_1^3w_2 + w_1^2w_3$ is in $\mathrm{H}^5(\mathrm{BO}, \mathbb{Z}) \otimes \mathbb{Z}_2$ While among the linear combinations of $w_1^6, w_1^2w_2^2, w_3^2, w_1^4w_2, w_1^3w_3, w_1w_2w_3$, only $w_1^2w_2^2$, $w_1^3w_3$, w_1^6, and $w_1^2w_2^2 + w_3^2$ are in $\mathrm{H}^6(\mathrm{BO}, \mathbb{Z}) \otimes \mathbb{Z}_2$.

So we claim that $\mathrm{Ker}d_4^{(2,3)}$ is spanned by $x\gamma$ where $\gamma = b_1w_1^2 + b_2w_2$ with $a_1b_1 = 0$, $a_1b_2 + a_2b_1 = a_3b_1$, $a_2b_2 = 0$, $a_3b_2 = 0$. While $\mathrm{Ker}d_4^{(3,3)}$ is spanned by $x\zeta$ where $\zeta = c_1w_1^3 + c_2w_1w_2 + c_3w_3$ with $a_1c_2 + c_1a_2 = 0$, $a_2c_3 + c_2a_3 = 0$.

In the following cases, we only consider the topological terms involving the cohomology classes of $\mathrm{B}^2\mathbb{Z}_2$.

Case 1: $\beta = 0$, there is no differential in Figure 15, the 5d topological terms are x_3w_2 (or x_2w_3), $x_3w_1^2$ (or $x_2w_1^3$) and x_2x_3 (see Figure 16). Note that $x_3w_2 = x_2w_3$, $x_3w_1^2 = x_2w_1^3$ and $x_5 = x_3(w_1^2 + w_2)$ by Wu formula (2.52). In this case $\mathrm{B}\mathbb{G} = \mathrm{BO} \times \mathrm{B}^2\mathbb{Z}_2$, $MT\mathbb{G} = M\mathrm{O} \wedge (\mathrm{B}^2\mathbb{Z}_2)_+$, $\pi_d MT\mathbb{G} = \Omega_d^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$. This case will be discussed later in another way.

Case 2: $\beta = w_1^3$, $a_1 = 1$, $a_2 = a_3 = 0$. $\mathrm{Ker}d_4^{(2,3)}$ is spanned by $x\gamma$ where $\gamma = b_1w_1^2 + b_2w_2$ with $b_1 = b_2 = 0$. $\mathrm{Ker}d_4^{(3,3)}$ is spanned by $x\zeta$ where $\zeta = c_1w_1^3 + c_2w_1w_2 + c_3w_3$ with $c_2 = 0$. At the position (3,3), $x_2w_1^3$ is killed in the E_3 page, x_2w_3 survives to the E_∞ page, so the 5d topological terms are x_2w_3 and x_2x_3.

Case 3: $\beta = w_1w_2$, $a_1 = 0$, $a_2 = 1$, $a_3 = 0$. $\mathrm{Ker}d_4^{(2,3)}$ is spanned by $x\gamma$ where $\gamma = b_1w_1^2 + b_2w_2$ with $b_1 = b_2 = 0$. $\mathrm{Ker}d_4^{(3,3)}$ is spanned by $x\zeta$ where $\zeta = c_1w_1^3 + c_2w_1w_2 + c_3w_3$ with $c_1 = c_3 = 0$. At the position (3,3), x_2w_3 and $x_2w_1^3$ are killed in the E_4 page, so the 5d topological term is x_2x_3.

Case 4: $\beta = w_3$, $a_1 = a_2 = 0$, $a_3 = 1$. $\mathrm{Ker}d_4^{(2,3)}$ is spanned by $x\gamma$ where $\gamma = b_1w_1^2 + b_2w_2$ with $b_1 = b_2 = 0$. $\mathrm{Ker}d_4^{(3,3)}$ is spanned by $x\zeta$ where $\zeta = c_1w_1^3 + c_2w_1w_2 + c_3w_3$ with $c_2 = 0$. At the position (3,3), x_2w_3 is killed in the E_3 page, $x_2w_1^3$ survives to the E_∞ page, so the 5d topological terms are $x_2w_1^3$ and x_2x_3.

Case 5: $\beta = w_1^3 + w_1w_2$, $a_1 = a_2 = 1$, $a_3 = 0$. $\mathrm{Ker}d_4^{(2,3)}$ is spanned by $x\gamma$ where $\gamma = b_1w_1^2 + b_2w_2$ with $b_1 = b_2 = 0$. $\mathrm{Ker}d_4^{(3,3)}$ is spanned by $x\zeta$ where $\zeta = c_1w_1^3 + c_2w_1w_2 + c_3w_3$ with $c_1 + c_2 = 0$, $c_3 = 0$. At the position (3,3), $x_2(w_1^3 + w_1w_2)$ is killed in the E_3 page, x_2w_3 is killed in the E_4 page, but since $x_2w_1w_2 = \mathrm{Sq}^3x_2 = 0$ by Wu formula (2.52), so the 5d topological term is x_2x_3.

Case 6: $\beta = w_1w_2+w_3$, $a_1 = 0$, $a_2 = a_3 = 1$. $\mathrm{Ker}d_4^{(2,3)}$ is spanned by $x\gamma$ where $\gamma = b_1w_1^2+b_2w_2$ with $b_2 = 0$. $\mathrm{Ker}d_4^{(3,3)}$ is spanned by $x\zeta$ where $\zeta = c_1w_1^3 + c_2w_1w_2 + c_3w_3$ with $c_1 = 0$, $c_2 + c_3 = 0$. At the position (3,3), $x_2(w_1w_2 + w_3)$ is killed in the E_3 page, $x_2w_1^3$ is killed in the E_4 page, but since $x_2w_1w_2 = \mathrm{Sq}^3x_2 = 0$ by Wu formula (2.52), so the 5d topological terms is x_2x_3.

Case 7: $\beta = w_1^3+w_3$, $a_1 = 1$, $a_2 = 0$, $a_3 = 1$. $\mathrm{Ker}d_4^{(2,3)}$ is spanned by $x\gamma$ where $\gamma = b_1w_1^2+b_2w_2$ with $b_1 = b_2 = 0$. $\mathrm{Ker}d_4^{(3,3)}$ is spanned by $x\zeta$ where $\zeta = c_1w_1^3 + c_2w_1w_2 + c_3w_3$ with $c_2 = 0$. At the position (3,3), $x_2(w_1^3 + w_3)$ is killed in the E_3 page, so the 5d topological terms are $x_2w_1^3 = x_2w_3$ and x_2x_3.

Case 8: $\beta = w_1^3 + w_1w_2 + w_3$, $a_1 = a_2 = a_3 = 1$. $\mathrm{Ker}d_4^{(2,3)}$ is spanned by $x\gamma$ where $\gamma = b_1w_1^2 + b_2w_2$ with $b_1 = b_2 = 0$. $\mathrm{Ker}d_4^{(3,3)}$ is spanned by $x\zeta$ where $\zeta = c_1w_1^3 + c_2w_1w_2 + c_3w_3$ with $c_1 + c_2 = 0$, $c_2 + c_3 = 0$. At the position (3,3), $x_2(w_1^3 + w_1w_2 + w_3)$ is killed in the E_3 page, but since $x_2w_1w_2 = \mathrm{Sq}^3x_2 = 0$ by Wu formula (2.52), so the 5d topological term are $x_2w_1^3 = x_2w_3$ and x_2x_3.

5 O/SO/Spin/Pin$^\pm$ bordism groups of classifying spaces

In this section, we compute the O/SO/Spin/Pin$^\pm$ bordism groups of the classifying space of the group $\mathbb{G} = G_a \times \mathrm{B}G_b$: $\mathrm{B}\mathbb{G} = \mathrm{B}G_a \times \mathrm{B}^2G_b$. Here $\mathrm{B}G_b$ is a group since G_b is abelian.

We briefly comment the difference between a previous cobordism theory [25] and this work: In all Adams charts of the computation in [25], there are no nonzero differentials, while in this paper we encounter nonzero differentials d_n due to the (p, p^n)-Bockstein homomorphisms in the computation involving $\mathrm{B}^2\mathbb{Z}_{p^n}$ and $\mathrm{B}\mathbb{Z}_{p^n}$.

5.1 Introduction

For $H = $ O/SO/Spin/Pin$^\pm$ and the group $H \times \mathbb{G}$, define

$$MT(H \times \mathbb{G}) := \mathrm{Thom}(\mathrm{B}(H \times \mathbb{G}); -V) \tag{5.1}$$

where V is the induced virtual bundle over $\mathrm{B}(H \times \mathbb{G})$ by the composition $\mathrm{B}(H \times \mathbb{G}) \to \mathrm{B}H \to \mathrm{BO}$ where the first map is the projection, the second map is the natural homomorphism.

By the Pontryagin-Thom isomorphism (1.10) and the property of Thom space (1.7), $\Omega_d^H(\mathrm{B}\mathbb{G}) = \pi_d(MTH \wedge \mathrm{B}\mathbb{G}_+) = \pi_d(MT(H \times \mathbb{G}))$. Hence we can define

$$\Omega_d^{H\times\mathbb{G}} := \pi_d(MT(H \times \mathbb{G})) = \Omega_d^H(\mathrm{B}\mathbb{G}). \tag{5.2}$$

$$\mathrm{TP}_n(H \times \mathbb{G}) := [MT(H \times \mathbb{G}), \Sigma^{n+1}I\mathbb{Z}] \tag{5.3}$$

Here X_+ is the disjoint union of X and a point. $MT\mathrm{O} = M\mathrm{O}$, $MT\mathrm{SO} = M\mathrm{SO}$, $MT\mathrm{Spin} = M\mathrm{Spin}$, $MT\mathrm{Pin}^+ = M\mathrm{Pin}^-$, $MT\mathrm{Pin}^- = M\mathrm{Pin}^+$. $\pi_d(\mathcal{B})$ is the d-th stable homotopy group of the spectrum $\mathcal{B}$.

$[\mathcal{B}, \Sigma^{n+1} I\mathbb{Z}]$ stands for the homotopy classes of maps from spectrum $\mathcal{B}$ to the $(n+1)$-th suspension of spectrum $I\mathbb{Z}$. The Anderson dual $I\mathbb{Z}$ is a spectrum that is the fiber of $I\mathbb{C} \to I\mathbb{C}^\times$ where $I\mathbb{C}(I\mathbb{C}^\times)$ is the Brown-Comenetz dual spectrum defined by

$$[X, I\mathbb{C}] = \mathrm{Hom}(\pi_0 X, \mathbb{C}), \tag{5.4}$$

$$[X, I\mathbb{C}^\times] = \mathrm{Hom}(\pi_0 X, \mathbb{C}^\times). \tag{5.5}$$

By the work of Freed-Hopkins [4], there is a 1:1 correspondence

$$\left\{ \begin{array}{c} \text{deformation classes of reflection positive} \\ \text{invertible } n\text{-dimensional extended topological} \\ \text{field theories with symmetry group } H_n \times \mathbb{G} \end{array} \right\} \cong [MT(H \times \mathbb{G}), \Sigma^{n+1} I\mathbb{Z}]_{\text{tors}}. \tag{5.6}$$

There is an exact sequence

$$0 \to \mathrm{Ext}^1(\pi_n \mathcal{B}, \mathbb{Z}) \to [\mathcal{B}, \Sigma^{n+1} I\mathbb{Z}] \to \mathrm{Hom}(\pi_{n+1}\mathcal{B}, \mathbb{Z}) \to 0 \tag{5.7}$$

for any spectrum $\mathcal{B}$, especially for $MT(H \times \mathbb{G})$. The torsion part $[MT(H \times \mathbb{G}), \Sigma^{n+1} I\mathbb{Z}]_{\text{tors}}$ is $\mathrm{Ext}^1((\pi_n MT(H \times \mathbb{G}))_{\text{tors}}, \mathbb{Z}) = \mathrm{Hom}((\pi_n MT(H \times \mathbb{G}))_{\text{tors}}, \mathrm{U}(1))$.

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[a] \tag{5.8}$$

where $|a| = 1$.

Theorem 18 (**Serre**, Ref. [67])**.**

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[\mathrm{Sq}^I x_2 | I \text{ admissible}, ex(I) < 2] = \mathbb{Z}_2[\mathrm{Sq}^{2^{i-1}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1 x_2 | i \geq 0] \tag{5.9}$$

where x_2 is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$.

Denote $\mathrm{Sq}^{2^{i-1}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1 x_2 = x_{2^i+1}$.

Here $\mathrm{Sq}^I = \mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots$ and $I = (i_1, i_2, \dots)$ is admissible if $i_s \geq 2i_{s+1}$ for $s \geq 1$, $ex(I) = \sum_{s\geq 1}(i_s - 2i_{s+1})$.

Theorem 19 (Ref. [67])**.**

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3, \mathbb{Z}_3) = F_{\mathbb{Z}_3}[a', b'] = \Lambda_{\mathbb{Z}_3}(a') \otimes \mathbb{Z}_3[b'] \tag{5.10}$$

where $|a'| = 1$ and $b' = \beta_{(3,3)} a'$.

Here $\beta_{(3,3)}$ is the Bockstein homomorphism in $\mathcal{A}_3$.

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3) = F_{\mathbb{Z}_3}[x_2', \beta_{(3,3)}x_2', Q_i x_2', \beta_{(3,3)}Q_i x_2', i \geq 1] = \mathbb{Z}_3[x_2', \beta_{(3,3)}Q_i x_2', i \geq 1] \otimes \Lambda_{\mathbb{Z}_3}(\beta_{(3,3)}x_2', Q_i x_2', i \geq 1)$$

where $|x_2'| = 2$ and Q_i is defined inductively by $Q_0 = \beta_{(3,3)}$, $Q_i = P^{3^{i-1}} Q_{i-1} - Q_{i-1} P^{3^{i-1}}$ for $i \geq 1$. Let $x_3' = \beta_{(3,3)} x_2'$, $x'_{2\cdot 3^i+1} = Q_i x_2'$, $x'_{2\cdot 3^i+2} = \beta_{(3,3)} Q_i x_2'$ for $i \geq 1$.

Here P^n is the n-th Steenrod power in $\mathcal{A}_3$.

$$\mathrm{H}^*(\mathrm{BPSU}(2),\mathbb{Z}_2)=\mathbb{Z}_2[w_2',w_3']. \tag{5.11}$$

$$\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2)=\mathbb{Z}_2[c_2,c_3]. \tag{5.12}$$

Here w_i' is the i-th Stiefel-Whitney class $w_i(\mathrm{PSU}(2))$ of the universal principal PSU(2)-bundle over BPSU(2). Let p_i' be the i-th Pontryagin class $p_i(\mathrm{PSU}(2))$ of the universal principal PSU(2)-bundle over BPSU(2), then $p_1'(\mod 2)=w_2'^2$.

c_i is the i-th Chern class $c_i(\mathrm{PSU}(3))$ of the universal principal PSU(3)-bundle over BPSU(3).

Since $\frac{\mathrm{SU}(3)\times\mathrm{U}(1)}{\mathbb{Z}_3}=\mathrm{U}(3)$, $\mathrm{PSU}(3)=\mathrm{PU}(3)$.

Theorem 20 (Ref. [68])**.**

$$\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_3)=F_{\mathbb{Z}_3}[z_2,z_3,z_7,z_8,z_{12}]/J \tag{5.13}$$

where $|z_i|=i$, $J=(z_2z_3,z_2z_7,z_2z_8+z_3z_7)$ is the ideal generated by $z_2z_3,z_2z_7,z_2z_8+z_3z_7$ and $z_3=\beta_{(3,3)}z_2$, $z_7=P^1z_3$, $z_8=\beta_{(3,3)}z_7$. Note that $c_2(\mod 3)=z_2^2$, $c_3(\mod 3)=z_2^3$.

In the following subsections, all bordism invariants are the pullback of cohomology classes along classifying maps $f:M\to X$ and $g:M\to \mathrm{B}H$.

5.2 Point

5.2.1 Ω_d^{O}

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t}=\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\pi_{t-s}(M\mathrm{O})_2^{\wedge}=\Omega_{t-s}^{\mathrm{O}}. \tag{5.14}$$

Here $\pi_{t-s}(M\mathrm{O})_2^{\wedge}$ is the 2-completion of the group $\pi_{t-s}(M\mathrm{O})$.

The mod 2 cohomology of Thom spectrum $M\mathrm{O}$ is

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)=\mathcal{A}_2\otimes\Omega^* \tag{5.15}$$

where $\Omega=\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]$ is the unoriented bordism ring, Ω^* is the $\mathbb{Z}_2$-linear dual of Ω.

On the other hand, $\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)=\mathbb{Z}_2[w_1,w_2,w_3,\dots]U$ where U is the Thom class of the virtual bundle (of dimension 0) over BO which is the colimit of E_n-n and E_n is the universal n-bundle over $\mathrm{BO}(n)$, w_i is the i-th Stiefel-Whitney class of the virtual bundle (of dimension 0) over BO. Note that the pullback of the virtual bundle (of dimension 0) over BO along the map $g:M\to\mathrm{BO}$ is just $TM-d$ where M is a d-dimensional manifold and TM is the tangent bundle of M, g is given by the O-structure on M. We will not distinguish w_i and $w_i(TM)$.

Here y_i are manifold generators, for example, $y_2 = \mathbb{RP}^2$, $y_4 = \mathbb{RP}^4$, y_5 is Wu manifold SU(3)/SO(3). By Thom's result [1], two manifolds are unorientedly bordant if and only if they have identical sets of Stiefel-Whitney characteristic numbers. The nonvanishing Stiefel-Whitney numbers of $y_2 = \mathbb{RP}^2$ are w_2 and w_1^2, the nonvanishing Stiefel-Whitney numbers of $y_2^2 = \mathbb{RP}^2 \times \mathbb{RP}^2$ are w_2^2 and w_4, the nonvanishing Stiefel-Whitney numbers of $y_4 = \mathbb{RP}^4$ are w_1^4 and w_4, the only nonvanishing Stiefel-Whitney number of Wu manifold SU(3)/SO(3) is w_2w_3.

So $y_2^* = w_1^2$ or w_2, $(y_2^2)^* = w_2^2$, $y_4^* = w_1^4$, $y_5^* = w_2w_3$, etc, where y_i^* is the $\mathbb{Z}_2$-linear dual of $y_i \in \Omega$.

Below we choose $y_2^* = w_1^2$ by default, this is reasonable since $\mathrm{Sq}^2(x_{d-2}) = (w_2 + w_1^2)x_{d-2}$ on d-manifold by Wu formula (2.52).

Hence we have the following theorem

i	Ω_i^{O}
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 21.

The bordism invariant of Ω_2^{O} is w_1^2.

The bordism invariants of Ω_4^{O} are w_1^4, w_2^2.

The bordism invariant of Ω_5^{O} is w_2w_3.

i	$\mathrm{TP}_i(\mathrm{O})$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 22.

The 2d topological term is w_1^2.

The 4d topological terms are w_1^4, w_2^2.

The 5d topological term is w_2w_3.

5.2.2 Ω_d^{SO}

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{SO})_2^{\wedge} = \Omega_{t-s}^{\mathrm{SO}}. \tag{5.16}$$

The mod 2 cohomology of Thom spectrum $M\mathrm{SO}$ is

$$\mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_2) = \mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \oplus \Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \oplus \Sigma^5\mathcal{A}_2 \oplus \cdots . \tag{5.17}$$

$$\cdots \longrightarrow \Sigma^3\mathcal{A}_2 \longrightarrow \Sigma^2\mathcal{A}_2 \longrightarrow \Sigma\mathcal{A}_2 \longrightarrow \mathcal{A}_2 \longrightarrow \mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1 \tag{5.18}$$

is an $\mathcal{A}_2$-resolution where the differentials d_1 are induced by Sq^1.

The E_2 page is shown in Figure 17.

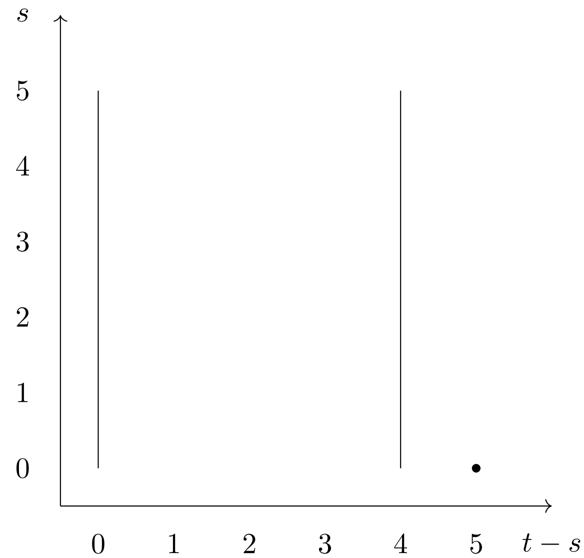


Figure 17: Ω_*^{SO}

Hence we have the following theorem

i	Ω_i^{SO}
0	$\mathbb{Z}$
1	0
2	0
3	0
4	$\mathbb{Z}$
5	$\mathbb{Z}_2$

Theorem 23.

The bordism invariant of Ω_4^{SO} is σ.

i	$\mathrm{TP}_i(\mathrm{SO})$
0	0
1	0
2	0
3	$\mathbb{Z}$
4	0
5	$\mathbb{Z}_2$

Here σ is the signature of a 4-manifold.

The bordism invariant of Ω_5^{SO} is w_2w_3.

Theorem 24.

Since $\sigma = \frac{p_1(TM)}{3}$, $p_1(TM) = \mathrm{dCS}_3^{(TM)}$, the 3d topological term is $\frac{1}{3}\mathrm{CS}_3^{(TM)}$.

The 5d topological term is w_2w_3.

5.2.3 Ω_d^{Spin}

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{Spin})_2^\wedge = \Omega_{t-s}^{\mathrm{Spin}}. \tag{5.19}$$

The mod 2 cohomology of Thom spectrum $M\mathrm{Spin}$ is

$$\mathrm{H}^*(M\mathrm{Spin}, \mathbb{Z}_2) = \mathcal{A}_2 \otimes_{\mathcal{A}_2(1)} \{\mathbb{Z}_2 \oplus M\} \tag{5.20}$$

where M is a graded $\mathcal{A}_2(1)$-module with the degree i homogeneous part $M_i = 0$ for $i < 8$. Here $\mathcal{A}_2(1)$ stands for the subalgebra of $\mathcal{A}_2$ generated by Sq^1 and Sq^2. For $t - s < 8$, we can identify the E_2-page with

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2).$$

The E_2 page is shown in Figure 18.

Hence we have the following theorem

i	Ω_i^{Spin}
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}$
5	0

Theorem 25.

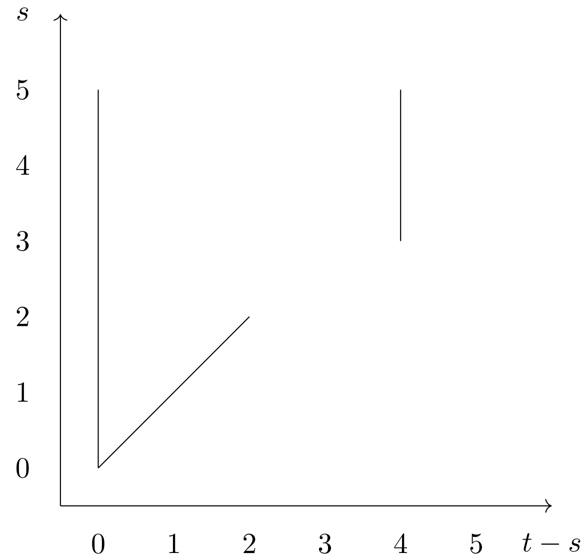


Figure 18: Ω_*^{Spin}

The bordism invariant of Ω_1^{Spin} is $\tilde{\eta}$.

Here $\tilde{\eta}$ is the "mod 2 index" of the 1d Dirac operator (#zero eigenvalues mod 2, no contribution from spectral asymmetry).

The bordism invariant of Ω_2^{Spin} is Arf (the Arf invariant).

The bordism invariant of Ω_4^{Spin} is $\frac{\sigma}{16}$.

i	$\mathrm{TP}_i(\mathrm{Spin})$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}$
4	0
5	0

Theorem 26.

The 1d topological term is $\tilde{\eta}$.

The 2d topological term is Arf.

The 3d topological term is $\frac{1}{48}\mathrm{CS}_3^{(TM)}$.

5.2.4 $\Omega_d^{\text{Pin}^+}$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(M\text{Pin}^-, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\text{Pin}^-)_2^\wedge = \Omega_{t-s}^{\text{Pin}^+}. \tag{5.21}$$

$M\text{Pin}^- = MT\text{Pin}^+ \sim M\text{Spin} \wedge S^1 \wedge MT\text{O}(1)$.

For $t - s < 8$, we can identify the E_2-page with

$$\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(\text{H}^{*-1}(MT\text{O}(1), \mathbb{Z}_2), \mathbb{Z}_2).$$

By Thom's isomorphism,

$$\text{H}^{*-1}(MT\text{O}(1), \mathbb{Z}_2) = \mathbb{Z}_2[w_1]U \tag{5.22}$$

where U is the Thom class of the virtual bundle $-E_1$ over BO(1), E_1 is the universal 1-bundle over BO(1) and w_1 is the 1st Stiefel-Whitney class of E_1 over BO(1). The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*-1}(MT\text{O}(1), \mathbb{Z}_2)$ and the E_2 page are shown in Figure 19, 20.

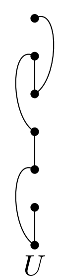


Figure 19: The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*-1}(MT\text{O}(1), \mathbb{Z}_2)$

Hence we have the following theorem

i	$\Omega_i^{\text{Pin}^+}$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_{16}$
5	0

Theorem 27.

The bordism invariant of $\Omega_2^{\text{Pin}^+}$ is $w_1 \cup \tilde{\eta}$.

The bordism invariant of $\Omega_3^{\text{Pin}^+}$ is $w_1 \cup \text{Arf}$.

The bordism invariant of $\Omega_4^{\text{Pin}^+}$ is η.

Here η is the usual Atiyah-Patodi-Singer eta-invariant of the 4d Dirac operator (="#zero eigenvalues + spectral asymmetry").

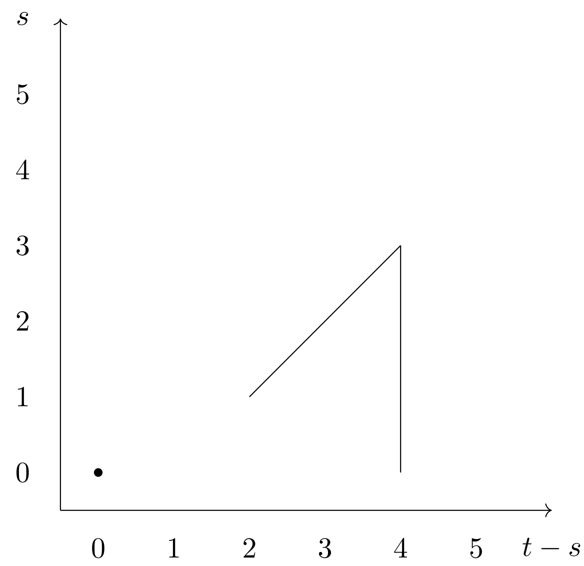


Figure 20: $\Omega_*^{\mathrm{Pin}^+}$

i	$\mathrm{TP}_i(\mathrm{Pin}^+)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_{16}$
5	0

Theorem 28.

The 2d topological term is $w_1 \cup \tilde{\eta}$.

The 3d topological term is $w_1 \cup \mathrm{Arf}$.

The 4d topological term is η.

5.2.5 $\Omega_d^{\mathrm{Pin}^-}$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{Pin}^+)_2^\wedge = \Omega_{t-s}^{\mathrm{Pin}^-}. \tag{5.23}$$

$M\mathrm{Pin}^+ = MT\mathrm{Pin}^- \sim M\mathrm{Spin} \wedge S^{-1} \wedge M\mathrm{O}(1)$.

For $t - s < 8$, we can identify the E_2-page with

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+1}(M\mathrm{O}(1), \mathbb{Z}_2), \mathbb{Z}_2).$$

By Thom's isomorphism,

$$\mathrm{H}^{*+1}(M\mathrm{O}(1), \mathbb{Z}_2) = \mathbb{Z}_2[w_1]U \tag{5.24}$$

where U is the Thom class of the universal 1-bundle E_1 over BO(1) and w_1 is the 1st Stiefel-Whitney class of E_1 over BO(1). The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1), \mathbb{Z}_2)$ and the E_2 page are shown in Figure 21, 22.



Figure 21: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1), \mathbb{Z}_2)$

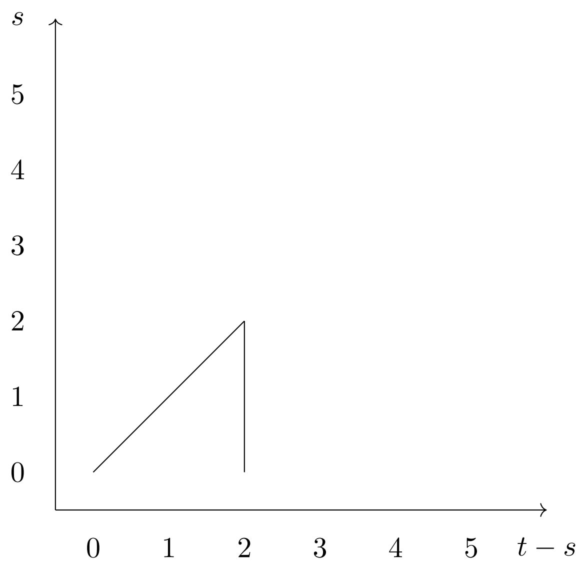


Figure 22: $\Omega_*^{\mathrm{Pin}^-}$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{Pin}^-}$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	0
5	0

Theorem 29.

The bordism invariant of $\Omega_1^{\mathrm{Pin}^-}$ is $\tilde{\eta}$.

The bordism invariant of $\Omega_2^{\text{Pin}^-}$ is ABK (the Arf-Brown-Kervaire invariant).

i	$\text{TP}_i(\text{Pin}^-)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	0
5	0

Theorem 30.

The 1d topological term is $\tilde{\eta}$.

The 2d topological term is ABK.

5.3 Atiyah-Hirzebruch spectral sequence

If $H = \text{O}/\text{SO}/\text{Spin}/\text{Pin}^{\pm}$, by the Atiyah-Hirzebruch spectral sequence, we have

$$\text{H}_p(\text{B}\mathbb{G}, \Omega_q^H) \Rightarrow \Omega_{p+q}^H(\text{B}\mathbb{G}). \tag{5.25}$$

If $H = \text{O}/\text{Pin}^{\pm}$, since Ω_d^H are finite, $\Omega_d^{H\times\mathbb{G}} = \Omega_d^H(\text{B}\mathbb{G})$ are also finite, so $\text{TP}_d(H \times \mathbb{G}) = \Omega_d^{H\times\mathbb{G}}$ for $H = \text{O}/\text{Pin}^{\pm}$.

If $H = \text{SO}/\text{Spin}$,

$$\Omega_q^{\text{SO}} = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q = 1 \\ 0 & q = 2 \\ 0 & q = 3 \\ \mathbb{Z} & q = 4 \\ \mathbb{Z}_2 & q = 5 \\ 0 & q = 6 \end{cases}. \tag{5.26}$$

$$\Omega_q^{\text{Spin}} = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}_2 & q = 1 \\ \mathbb{Z}_2 & q = 2 \\ 0 & q = 3 \\ \mathbb{Z} & q = 4 \\ 0 & q = 5 \\ 0 & q = 6 \end{cases}. \tag{5.27}$$

If $\text{H}_p(\text{B}\mathbb{G}, \mathbb{Z})$ are finite for $p > 0$, then $\Omega_6^H(\text{B}\mathbb{G})$ is finite and $\text{TP}_5(H \times \mathbb{G}) = \Omega_5^H(\text{B}\mathbb{G})$ for $H = \text{SO}/\text{Spin}$.

If $\mathbb{G} = \mathrm{PSU}(2) = \mathrm{SO}(3)$, since $\mathrm{H}_2(\mathrm{BSO}(3), \mathbb{Z})$ and $\mathrm{H}_6(\mathrm{BSO}(3), \mathbb{Z})$ are finite, $\Omega_6^H(\mathrm{B}\mathbb{G})$ is also finite and $\mathrm{TP}_5(H \times \mathbb{G}) = \Omega_5^H(\mathrm{B}\mathbb{G})$ for $H = \mathrm{SO}/\mathrm{Spin}$.

If $\mathbb{G} = \mathrm{PSU}(3)$, then $\mathrm{H}_6(\mathrm{BPSU}(3), \mathbb{Z})$ contains a $\mathbb{Z}$ while $\mathrm{H}_2(\mathrm{BPSU}(3), \mathbb{Z})$ does not, so $\Omega_6^H(\mathrm{B}\mathbb{G})$ contains a $\mathbb{Z}$ and $\mathrm{TP}_5(H \times \mathbb{G}) = \Omega_5^H(\mathrm{B}\mathbb{G}) \times \mathbb{Z}$ for $H = \mathrm{SO}/\mathrm{Spin}$.

5.4 $\mathrm{B}^2 G_b : \mathrm{B}^2\mathbb{Z}_2, \mathrm{B}^2\mathbb{Z}_3$

5.4.1 $\Omega_d^{\mathbf{O}}(\mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O} \wedge (\mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{O} \wedge (\mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2). \tag{5.28}$$

Theorem 31 (Thom)**.** • $\pi_* M\mathrm{O} = \Omega = \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]$.
• $\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) = \mathcal{A}_2 \otimes \Omega^*$ where Ω^* is the $\mathbb{Z}_2$-linear dual of Ω.

$y_2 = \mathbb{RP}^2$, $y_4 = \mathbb{RP}^4$, y_5 is the Wu manifold $\mathrm{W} = \mathrm{SU}(3)/\mathrm{SO}(3)$, ...
$y_2^* = w_2(TM)$ or $w_1(TM)^2$, $y_2^* = w_2(TM)^2$, $y_4^* = w_1(TM)^4$, $y_5^* = w_2(TM)w_3(TM)$, ...

Theorem 32 (Serre)**.**

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots]$$

where $x_3 = \mathrm{Sq}^1 x_2$, $x_5 = \mathrm{Sq}^2 x_3$, $x_9 = \mathrm{Sq}^4 x_5$, ...

By Künneth theorem,

$$\begin{aligned}
& \mathrm{H}^*(M\mathrm{O} \wedge (\mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2) \\
= \ & \mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) \\
= \ & \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \otimes \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots] \\
= \ & \mathcal{A}_2 \oplus 2\Sigma^2\mathcal{A}_2 \oplus \Sigma^3\mathcal{A}_2 \oplus 4\Sigma^4\mathcal{A}_2 \oplus 4\Sigma^5\mathcal{A}_2 \oplus \cdots
\end{aligned} \tag{5.29}$$

Here $\Sigma^n\mathcal{A}_2$ is the n-th iterated shift of the graded algebra $\mathcal{A}_2$.

Since

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\Sigma^r\mathcal{A}_2, \mathbb{Z}_2) = \begin{cases} \mathrm{Hom}_{\mathcal{A}_2}^t(\Sigma^r\mathcal{A}_2, \mathbb{Z}_2) = \mathbb{Z}_2 & \text{if } t = r, s = 0 \\ 0 & \text{else} \end{cases}, \tag{5.30}$$

we have the following theorem

Theorem 33.

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^4$

The bordism invariants of $\Omega_2^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ are x_2, w_1^2.

The bordism invariant of $\Omega_3^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ is $x_3 = w_1x_2$.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ are $x_2^2, w_1^4, w_1^2x_2, w_2^2$.

The bordism invariants of $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ are $x_2x_3, x_5, w_1^2x_3, w_2w_3$.

Here w_i is the i-th Stiefel-Whitney class of the tangent bundle of M, x_2 is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \mathrm{Sq}^1x_2$, $x_5 = \mathrm{Sq}^2\mathrm{Sq}^1x_2$, since there is a map $f : M \to \mathrm{B}^2\mathbb{Z}_2$ in the definition of cobordism group, we identify x_2 with $f^*(x_2) = f$. By Wu formula, $\mathrm{Sq}^2x_{d-2} = (w_2 + w_1^2)x_{d-2}$ on d-manifolds.

i	$\mathrm{TP}_i(\mathrm{O} \times \mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^4$

Theorem 34.

The 2d topological terms are x_2, w_1^2.

The 3d topological term is $x_3 = w_1x_2$.

The 4d topological terms are $x_2^2, w_1^4, w_1^2x_2, w_2^2$.

The 5d topological terms are $x_2x_3, x_5, w_1^2x_3, w_2w_3$.

5.4.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO} \wedge (\mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{SO} \wedge (\mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2). \quad (5.31)$$

Since $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots]$ where x_2 is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \mathrm{Sq}^1x_2$, $x_5 = \mathrm{Sq}^2\mathrm{Sq}^1x_2$, $x_9 = \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1x_2$, etc, $\mathrm{Sq}^1x_2 = x_3$, $\mathrm{Sq}^1x_3 = 0$, $\mathrm{Sq}^1(x_2^2) = 0$, $\mathrm{Sq}^1(x_2x_3) = \mathrm{Sq}^1(x_5) = x_3^2$. We have used (2.50) and the Adem relations (2.67).

We shift Figure 2 the same times as the dimension of $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ at each degree as a $\mathbb{Z}_2$-vector space. We obtain the E_1 page for $\Omega_*^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$, the differentials d_1 are induced by Sq^1, as shown in Figure 23.

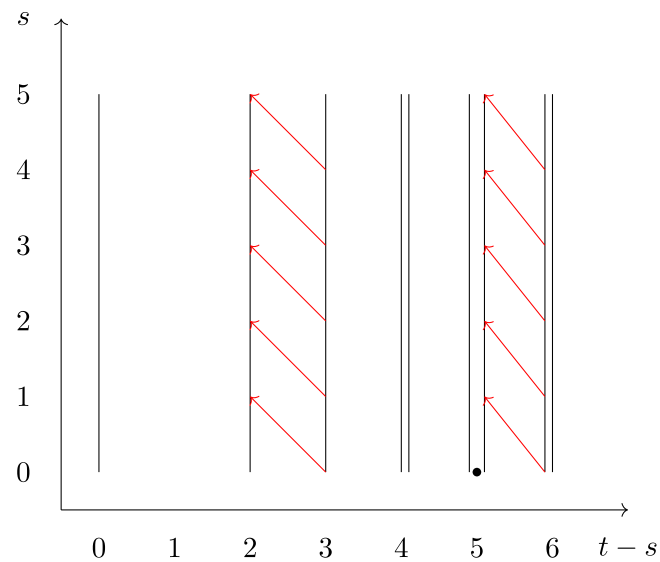


Figure 23: E_1 page for $\Omega_*^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$

There is a differential d_2 corresponding to the Bockstein homomorphism $\beta_{(2,4)}:\mathrm{H}^*(-,\mathbb{Z}_4)\to\mathrm{H}^{*+1}(-,\mathbb{Z}_2)$ associated to $0\to\mathbb{Z}_2\to\mathbb{Z}_8\to\mathbb{Z}_4\to 0$ [57]. See 2.5 for the definition of Bockstein homomorphisms. Note that

$$
\begin{aligned}
\beta_{(2,4)}\mathcal{P}_2(x_2) &= \frac{1}{4}\delta\mathcal{P}_2(x_2) \mod 2\\
&= \frac{1}{4}\delta(x_2\cup x_2+x_2\underset{1}{\cup}\delta x_2)\\
&= \frac{1}{4}(\delta x_2\cup x_2+x_2\cup\delta x_2+\delta(x_2\underset{1}{\cup}\delta x_2))\\
&= \frac{1}{4}(2x_2\cup\delta x_2+\delta x_2\underset{1}{\cup}\delta x_2)\\
&= x_2\cup(\frac{1}{2}\delta x_2)+(\frac{1}{2}\delta x_2)\underset{1}{\cup}(\frac{1}{2}\delta x_2)\\
&= x_2\mathrm{Sq}^1x_2+\mathrm{Sq}^1x_2\underset{1}{\cup}\mathrm{Sq}^1x_2\\
&= x_2\mathrm{Sq}^1x_2+\mathrm{Sq}^2\mathrm{Sq}^1x_2\\
&= x_2x_3+x_5
\end{aligned}
\tag{5.32}
$$

We have used $\beta_{(2,4)}=\frac{1}{4}\delta \mod 2$, the Steenrod's formula (2.12), $\mathrm{Sq}^1=\beta_{(2,2)}=\frac{1}{2}\delta \mod 2$, and the definition $\mathrm{Sq}^k x_n = x_n \underset{n-k}{\cup} x_n$.

So there is a differential such that $d_2(x_2x_3+x_5)=x_2^2h_0^2$.

Then take the differentials d_2 into account, we obtain the E_2 page for $\Omega_*^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$, as shown in Figure 24.

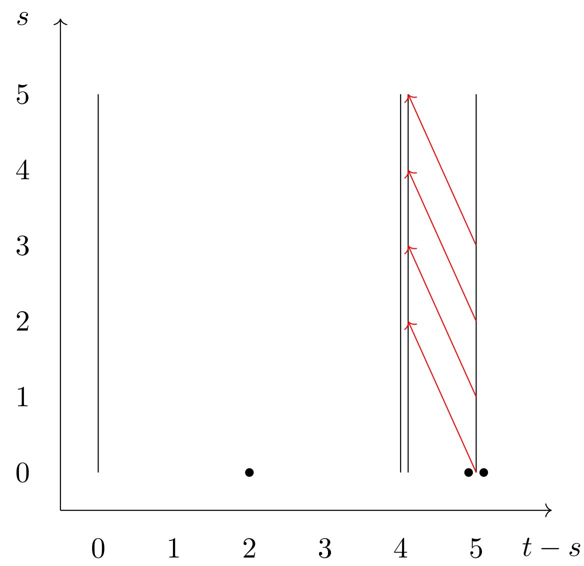


Figure 24: $\Omega_*^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}\times\mathbb{Z}_4$
5	$\mathbb{Z}_2^2$

Theorem 35.

The bordism invariant of $\Omega_2^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$ is x_2.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$ are σ and $\mathcal{P}_2(x_2)$.

The bordism invariants of $\Omega_5^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$ are $x_5 = x_2x_3$ and w_2w_3.

Here $\mathcal{P}_2(x_2)$ is the Pontryagin square of x_2. σ is the signature of a 4-manifold M. $x_2x_3 + x_5 = \frac{1}{2}\tilde{w}_1\mathcal{P}_2(x_2)$ [69] where $\tilde{w}_1$ is the twisted first Stiefel-Whitney class of the tangent bundle, in particular, $w_1 = 0$ implies $\tilde{w}_1 = 0$, so $x_2x_3 = x_5$ on oriented 5-manifolds.

i	$\mathrm{TP}_i(\mathrm{SO}\times\mathrm{B}\mathbb{Z}_2)$
0	0
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}$
4	$\mathbb{Z}_4$
5	$\mathbb{Z}_2^2$

Theorem 36.

The 2d topological term is x_2.

The 3d topological term is $\frac{1}{3}\mathrm{CS}_3^{(TM)}$.

The 4d topological term is $\mathcal{P}_2(x_2)$.

The 5d topological terms are $x_5 = x_2x_3$ and w_2w_3.

5.4.3 $\Omega_d^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin} \wedge (\mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{Spin} \wedge (\mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_2). \quad (5.33)$$

For $t-s<8$, we can identify the E_2-page with

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2).$$

$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots]$ where x_2 is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \mathrm{Sq}^1x_2$, $x_5 = \mathrm{Sq}^2\mathrm{Sq}^1x_2$, $x_9 = \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1x_2$, etc, $\mathrm{Sq}^1x_2 = x_3$, $\mathrm{Sq}^2x_2 = x_2^2$, $\mathrm{Sq}^1x_3 = 0$, $\mathrm{Sq}^2x_3 = x_5$, $\mathrm{Sq}^1(x_2^2) = 0$, $\mathrm{Sq}^1(x_2x_3) = x_3^2$, $\mathrm{Sq}^1x_5 = \mathrm{Sq}^2x_2^2 = x_3^2$, $\mathrm{Sq}^2x_5 = 0$. $\mathrm{Sq}^2(x_2x_3) = x_2^2x_3 + x_2x_5$. We have used (2.50) and the Adem relations (2.67).

There is a differential d_2 corresponding to the Bockstein homomorphism $\beta_{(2,4)} : \mathrm{H}^*(-, \mathbb{Z}_4) \to \mathrm{H}^{*+1}(-, \mathbb{Z}_2)$ associated to $0 \to \mathbb{Z}_2 \to \mathbb{Z}_8 \to \mathbb{Z}_4 \to 0$ [57]. See 2.5 for the definition of Bockstein homomorphisms.

By (5.32), there is a differential such that $d_2(x_2x_3 + x_5) = x_2^2h_0^2$.

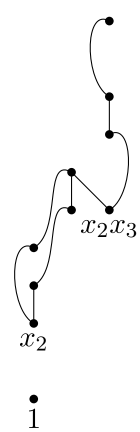


Figure 25: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ is shown in Figure 25. The dot at the bottom is a $\mathbb{Z}_2$ which has been discussed before. Now we consider the part above the bottom dot. We will use Lemma 11 several times. Two steps are shown in Figure 26, 27.

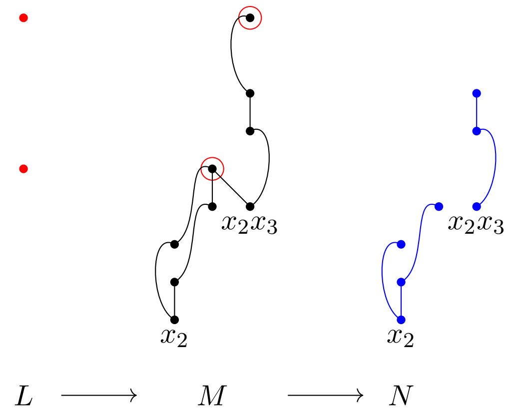


Figure 26: First step to get the E_2 page of $\Omega_*^{\text{Spin}}(\mathrm{B}^2\mathbb{Z}_2)$.

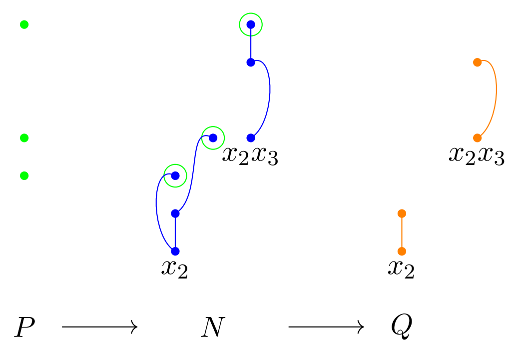


Figure 27: Second step to get the E_2 page of $\Omega_*^{\text{Spin}}(\mathrm{B}^2\mathbb{Z}_2)$.

We will proceed in the reversed order.

First, we apply Lemma 11 to the short exact sequence of $\mathcal{A}_2(1)$-modules: $0 \to P \to N \to Q \to 0$ in the second step (as shown in Figure 27), the Adams chart of $\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(N, \mathbb{Z}_2)$ is shown in Figure 28.

Next, we apply Lemma 11 to the short exact sequence of $\mathcal{A}_2(1)$-modules: $0 \to L \to M \to N \to 0$ in the first step (as shown in Figure 26), the Adams chart of $\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(M, \mathbb{Z}_2)$ is shown in Figure 29.

Then take the differentials d_2 into account, we obtain the E_2 page for $\Omega_*^{\text{Spin}}(\mathrm{B}^2\mathbb{Z}_2)$, as shown in Figure 30.

Hence we have the following theorem

Theorem 37.

The bordism invariants of $\Omega_2^{\text{Spin}}(\mathrm{B}^2\mathbb{Z}_2)$ are x_2 and Arf.

The bordism invariants of $\Omega_4^{\text{Spin}}(\mathrm{B}^2\mathbb{Z}_2)$ are $\frac{\sigma}{16}$ and $\frac{\mathcal{P}_2(x_2)}{2}$.

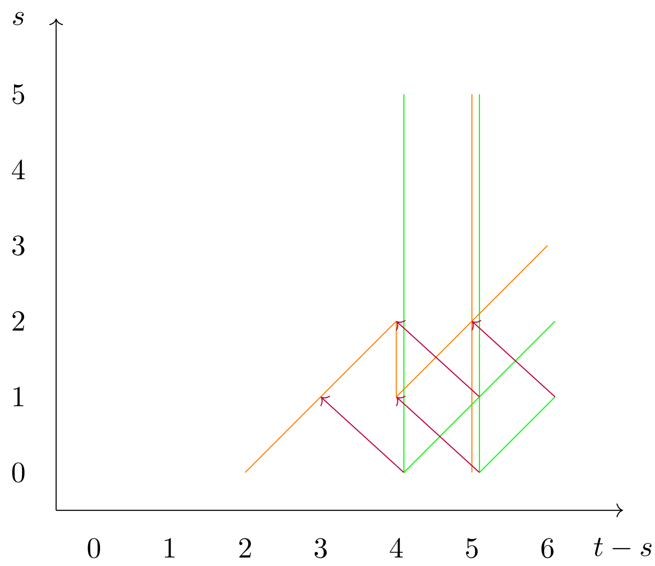


Figure 28: Adams chart of $\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(N, \mathbb{Z}_2)$. The arrows indicate the differential d_1.

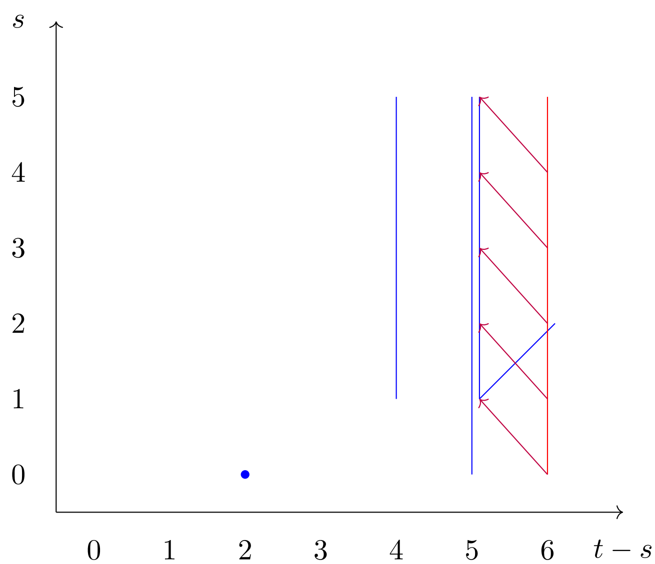


Figure 29: Adams chart of $\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(M, \mathbb{Z}_2)$. The arrows indicate the differential d_1.

i	$\Omega_i^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	0
4	$\mathbb{Z} \times \mathbb{Z}_2$
5	0

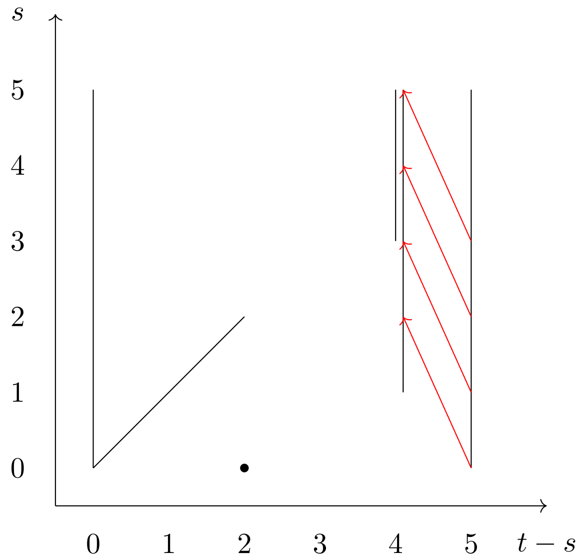


Figure 30: $\Omega_*^{\mathrm{Spin}}(\mathrm{B}^2\mathbb{Z}_2)$

By Wu formula, $x_2^2 = \mathrm{Sq}^2(x_2) = (w_2(TM) + w_1(TM)^2)x_2 = 0$ on Spin 4-manifolds, $x_5 = \mathrm{Sq}^2(x_3) = (w_2(TM) + w_1(TM)^2)x_3 = 0$ on Spin 5-manifolds, $\mathcal{P}_2(x_2) = x_2^2 = 0 \mod 2$ on Spin 4-manifolds.

Here Arf is the Arf invariant. σ is the signature of Spin 4-manifold, it is a multiple of 16 by Rokhlin's theorem.

i	$\mathrm{TP}_i(\mathrm{Spin} \times \mathrm{B}\mathbb{Z}_2)$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}$
4	$\mathbb{Z}_2$
5	0

Theorem 38.

The 2d topological terms are x_2 and Arf.

The 3d topological term is $\frac{1}{48}\mathrm{CS}_3^{(TM)}$.

The 4d topological term is $\frac{\mathcal{P}_2(x_2)}{2}$.

5.4.4 $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^- \wedge (\mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{Pin}^- \wedge (\mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_2). \quad (5.34)$$

$M\mathrm{Pin}^- = MT\mathrm{Pin}^+ \sim M\mathrm{Spin} \wedge S^1 \wedge MT\mathrm{O}(1)$.

For $t-s<8$, we can identify the E_2-page with

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_2(1)}(\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2).$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ and the E_2 page are shown in Figure 31, 32.

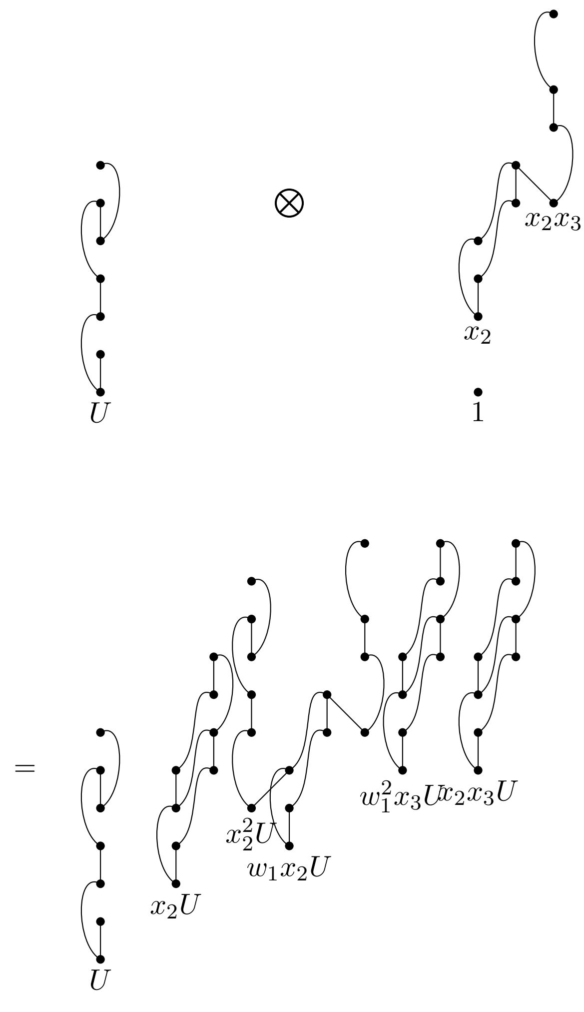


Figure 31: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$

Hence we have the following theorem

Theorem 39.

The bordism invariants of $\Omega_2^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_2)$ are x_2 and $w_1\tilde{\eta}$.

The bordism invariants of $\Omega_3^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_2)$ are $w_1x_2=x_3$ and $w_1\mathrm{Arf}$.

The bordism invariants of $\Omega_4^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_2)$ are $q_s(x_2)$ and η.

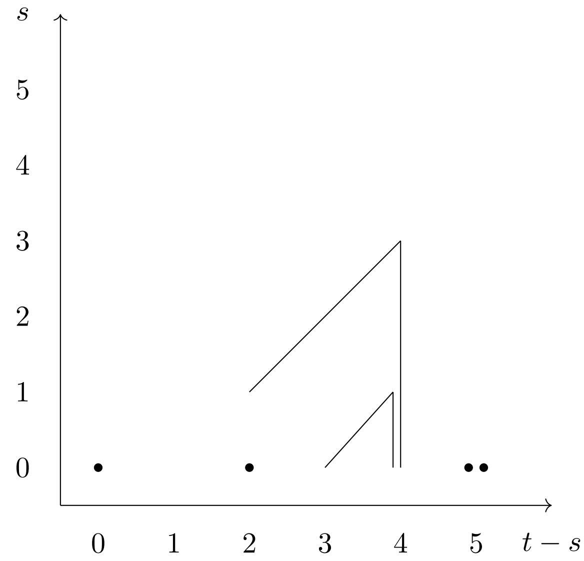


Figure 32: $\Omega_*^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_4\times\mathbb{Z}_{16}$
5	$\mathbb{Z}_2^2$

The bordism invariants of $\Omega_5^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_2)$ are x_2x_3 and $w_1^2x_3(=x_5)$.

Here $\tilde\eta$ is the "mod 2 index" of the 1d Dirac operator (#zero eigenvalues mod 2, no contribution from spectral asymmetry).

$x_3=\mathrm{Sq}^1x_2=w_1x_2$ on 3-manifolds by Wu formula.

q_s is explained in the footnotes of Table 3.

η is the usual Atiyah-Patodi-Singer eta-invariant of the 4d Dirac operator (="#zero eigenvalues + spectral asymmetry").

$x_5=\mathrm{Sq}^2x_3=(w_2+w_1^2)x_3=w_1^2x_3$ on Pin^+ 5-manifolds by Wu formula.

i	$\mathrm{TP}_i(\mathrm{Pin}^+\times\mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_4\times\mathbb{Z}_{16}$
5	$\mathbb{Z}_2^2$

Theorem 40.

The 2d topological terms are x_2 and $w_1\tilde{\eta}$.

The 3d topological terms are $w_1x_2 = x_3$ and $w_1\mathrm{Arf}$.

The 4d topological terms are $q_s(x_2)$ and η.

The 5d topological terms are x_2x_3 and $w_1^2x_3(= x_5)$.

5.4.5 $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+ \wedge (\mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \pi_{t-s}(M\mathrm{Pin}^+ \wedge (\mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2). \quad (5.35)$$

$M\mathrm{Pin}^+ = MT\mathrm{Pin}^- \sim M\mathrm{Spin} \wedge S^{-1} \wedge M\mathrm{O}(1)$.

For $t - s < 8$, we can identify the E_2-page with

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+1}(M\mathrm{O}(1), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2).$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ and the E_2 page are shown in Figure 33, 34.

Hence we have the following theorem

i	$\Omega_i^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2 \times \mathbb{Z}_8$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2$
5	$\mathbb{Z}_2$

Theorem 41.

The bordism invariants of $\Omega_2^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$ are x_2 and ABK.

The bordism invariant of $\Omega_3^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$ is $w_1x_2 = x_3$.

The bordism invariant of $\Omega_4^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$ is $w_1^2x_2$.

The bordism invariant of $\Omega_5^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$ is x_2x_3.

Here ABK is the Arf-Brown-Kervaire invariant. $x_3 = \mathrm{Sq}^1x_2 = w_1x_2$ on 3-manifolds by Wu formula.

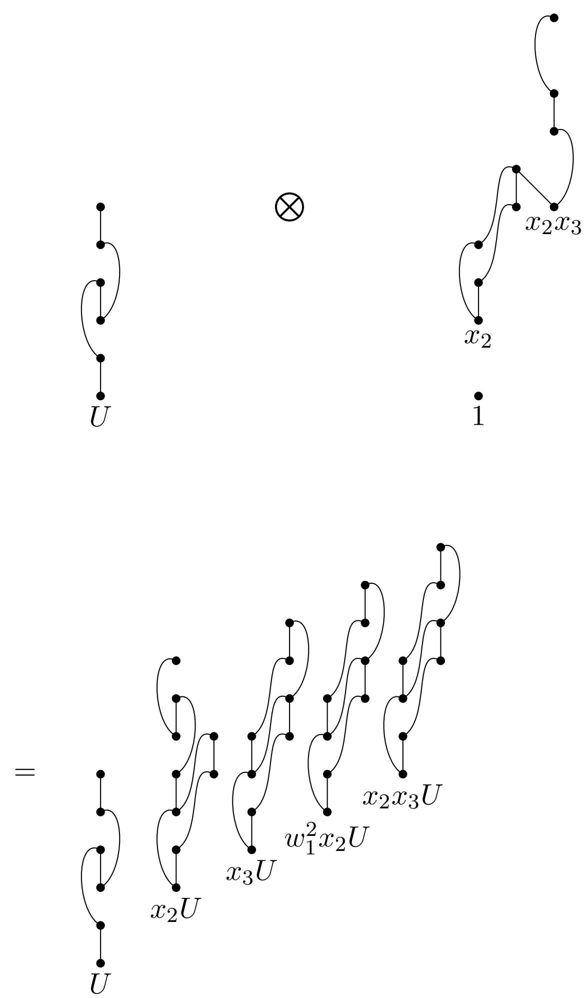


Figure 33: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$

i	$\mathrm{TP}_i(\mathrm{Pin}^-\times\mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2\times\mathbb{Z}_8$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2$
5	$\mathbb{Z}_2$

Theorem 42.

The 2d topological terms are x_2 and ABK.

The 3d topological term is $w_1x_2=x_3$.

The 4d topological term is $w_1^2x_2$.

The 5d topological term is x_2x_3.

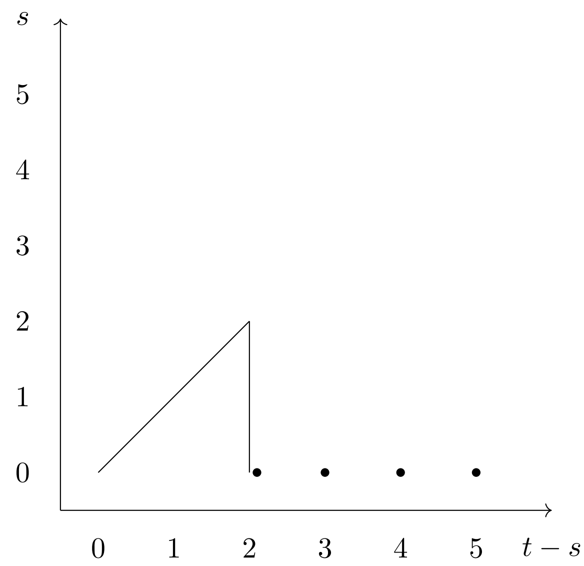


Figure 34: $\Omega_*^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_2)$

5.4.6 $\Omega_d^{\mathbf{O}}(\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O}\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}. \tag{5.36}$$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{O}\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}. \tag{5.37}$$

Since $M\mathrm{O}$ is the wedge sum of suspensions of the Eilenberg-MacLane spectrum $H\mathbb{Z}_2$, $\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_3)=0$, thus $\Omega_d^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}=0$.

Since $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_2$, we have $\Omega_d^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}=\Omega_d^{\mathrm{O}}$.

Hence $\Omega_d^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)=\Omega_d^{\mathrm{O}}$.

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 43.

The bordism invariant of $\Omega_2^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)$ is w_1^2.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)$ are w_1^4, w_2^2.

The bordism invariant of $\Omega_5^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_3)$ is w_2w_3.

i	$\mathrm{TP}_i(\mathrm{O}\times \mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 44.

The 2d topological term is w_1^2.

The 4d topological terms are w_1^4, w_2^2.

The 5d topological term is w_2w_3.

5.4.7 $\Omega_d^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}. \tag{5.38}$$

Since $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_2$, we have $\Omega_d^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}=\Omega_d^{\mathrm{SO}}$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}. \tag{5.39}$$

The dual of $\mathcal{A}_3=\mathrm{H}^*(H\mathbb{Z}_3,\mathbb{Z}_3)$ is

$$\mathcal{A}_{3*}=\mathrm{H}_*(H\mathbb{Z}_3,\mathbb{Z}_3)=\Lambda_{\mathbb{Z}_3}(\tau_0,\tau_1,\dots)\otimes\mathbb{Z}_3[\xi_1,\xi_2,\dots] \tag{5.40}$$

where $\tau_i=(P^{3^{i-1}}\cdots P^3P^1\beta_{(3,3)})^*$ and $\xi_i=(P^{3^{i-1}}\cdots P^3P^1)^*$. Let $C=\mathbb{Z}_3[\xi_1,\xi_2,\dots]\subseteq\mathcal{A}_{3*}$, then

$$\mathrm{H}_*(M\mathrm{SO},\mathbb{Z}_3)=C\otimes\mathbb{Z}_3[z_1',z_2',\dots] \tag{5.41}$$

where $|z_k'|=4k$ for $k\neq\frac{3^t-1}{2}$.

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_3)=(\mathbb{Z}_3[z_1',z_2',\dots])^*\otimes C^*=C^*\oplus\Sigma^8C^*\oplus\cdots \tag{5.42}$$

where $C^*=\mathcal{A}_3/(\beta_{(3,3)})$ and $(\beta_{(3,3)})$ is the two-sided ideal of $\mathcal{A}_3$ generated by $\beta_{(3,3)}$.

$$\cdots\longrightarrow\Sigma^2\mathcal{A}_3\oplus\Sigma^6\mathcal{A}_3\oplus\cdots\longrightarrow\Sigma\mathcal{A}_3\oplus\Sigma^5\mathcal{A}_3\oplus\cdots\longrightarrow\mathcal{A}_3\longrightarrow\mathcal{A}_3/(\beta_{(3,3)}) \tag{5.43}$$

is an $\mathcal{A}_3$-resolution of $\mathcal{A}_3/(\beta_{(3,3)})$ where the differentials d_1 are induced by $\beta_{(3,3)}$..

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_3)=\mathbb{Z}_3[x_2',x_8',\dots]\otimes\Lambda_{\mathbb{Z}_3}(x_3',x_7',\dots) \tag{5.44}$$

$\beta_{(3,3)}x_2'=x_3'$, $\beta_{(3,3)}x_2'^2=2x_2'x_3'$.

The E_2 page is shown in Figure 35.

Hence we have the following

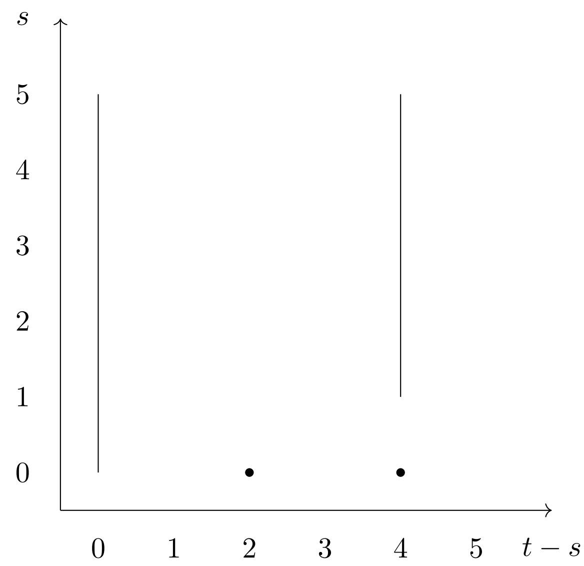


Figure 35: $\Omega_*^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)_3^\wedge$

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_3$
3	0
4	$\mathbb{Z}\times\mathbb{Z}_3$
5	$\mathbb{Z}_2$

Theorem 45.

The bordism invariant of $\Omega_2^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)$ is x_2'.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)$ are σ and $x_2'^2$.

The bordism invariant of $\Omega_5^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_3)$ is w_2w_3.

i	$\mathrm{TP}_i(\mathrm{SO}\times\mathrm{B}\mathbb{Z}_3)$
0	0
1	0
2	$\mathbb{Z}_3$
3	$\mathbb{Z}$
4	$\mathbb{Z}_3$
5	$\mathbb{Z}_2$

Theorem 46.

The 2d topological term is x_2'.

The 3d topological term is $\frac{1}{3}\mathrm{CS}_3^{(TM)}$.

The 4d topological term is $x_2'^2$.

The 5d topological term is w_2w_3.

5.4.8 $\Omega_d^{\text{Spin}}(\text{B}^2\mathbb{Z}_3)$

$$\text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(M\text{Spin}\wedge(\text{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Spin}}(\text{B}^2\mathbb{Z}_3)_2^\wedge. \tag{5.45}$$

Since $\text{H}^*(\text{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_2$, we have $\Omega_d^{\text{Spin}}(\text{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\text{Spin}}$.

$$\text{Ext}_{\mathcal{A}_3}^{s,t}(\text{H}^*(M\text{Spin}\wedge(\text{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\text{Spin}}(\text{B}^2\mathbb{Z}_3)_3^\wedge. \tag{5.46}$$

Since there is a short exact sequence of groups

$$1\longrightarrow\mathbb{Z}_2\longrightarrow\text{Spin}\longrightarrow\text{SO}\longrightarrow 1, \tag{5.47}$$

we have a fibration

$$\begin{array}{ccc}\text{B}\mathbb{Z}_2 & \longrightarrow & \text{BSpin}\\ & & \downarrow\\ & & \text{BSO}\end{array} \tag{5.48}$$

Take the localization at prime 3, we have a homotopy equivalence $\text{BSpin}_{(3)}\sim\text{BSO}_{(3)}$ since the localization of $\text{B}\mathbb{Z}_2$ at 3 is trivial. Take the Thom spectra, we have a homotopy equivalence $M\text{Spin}_{(3)}\sim M\text{SO}_{(3)}$. Hence

$$\text{H}^*(M\text{Spin},\mathbb{Z}_3)=\text{H}^*(M\text{SO},\mathbb{Z}_3). \tag{5.49}$$

We have the following

i	$\Omega_i^{\text{Spin}}(\text{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2\times\mathbb{Z}_3$
3	0
4	$\mathbb{Z}\times\mathbb{Z}_3$
5	0

Theorem 47.

The bordism invariants of $\Omega_2^{\text{Spin}}(\text{B}^2\mathbb{Z}_3)$ are Arf and x_2'.

The bordism invariants of $\Omega_4^{\text{Spin}}(\text{B}^2\mathbb{Z}_3)$ are $\frac{\sigma}{16}$ and $x_2'^2$.

Theorem 48.

The 2d topological terms are Arf and x_2'.

The 3d topological term is $\frac{1}{48}\text{CS}_3^{(TM)}$.

The 4d topological term is $x_2'^2$.

i	$\mathrm{TP}_i(\mathrm{Spin}\times \mathrm{B}\mathbb{Z}_3)$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2\times\mathbb{Z}_3$
3	$\mathbb{Z}$
4	$\mathbb{Z}_3$
5	0

5.4.9 $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^-\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{5.50}$$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+\wedge(\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{5.51}$$

Since $MT\mathrm{Pin}^+ = M\mathrm{Pin}^- \sim M\mathrm{Spin}\wedge S^1\wedge MT\mathrm{O}(1)$ and $\mathrm{H}^*(MT\mathrm{O}(1),\mathbb{Z}_3)=0$, we have $\mathrm{H}^*(M\mathrm{Pin}^-,\mathbb{Z}_3)=0$, thus $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)_3^\wedge=0$.

Since $\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_2$, we have $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\mathrm{Pin}^+}$.

Hence $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)=\Omega_d^{\mathrm{Pin}^+}$.

i	$\Omega_i^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_{16}$
5	0

Theorem 49.

The bordism invariant of $\Omega_2^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)$ is $w_1\tilde{\eta}$.

The bordism invariant of $\Omega_3^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)$ is $w_1\mathrm{Arf}$.

The bordism invariant of $\Omega_4^{\mathrm{Pin}^+}(\mathrm{B}^2\mathbb{Z}_3)$ is η.

i	$\mathrm{TP}_i(\mathrm{Pin}^+\times \mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_{16}$
5	0

Theorem 50.

The 2d topological term is $w_1\tilde{\eta}$.

The 3d topological term is $w_1\text{Arf}$.

The 4d topological term is η.

5.4.10 $\Omega_d^{\text{Pin}^-}(\text{B}^2\mathbb{Z}_3)$

$$\text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(M\text{Pin}^+\wedge(\text{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Pin}^-}(\text{B}^2\mathbb{Z}_3)_2^{\wedge}. \tag{5.52}$$

$$\text{Ext}_{\mathcal{A}_3}^{s,t}(\text{H}^*(M\text{Pin}^+\wedge(\text{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\text{Pin}^-}(\text{B}^2\mathbb{Z}_3)_3^{\wedge}. \tag{5.53}$$

Since $MT\text{Pin}^- = M\text{Pin}^+ \sim M\text{Spin}\wedge S^{-1}\wedge M\text{O}(1)$ and $\text{H}^*(M\text{O}(1),\mathbb{Z}_3) = 0$, we have $\text{H}^*(M\text{Pin}^+,\mathbb{Z}_3)=0$, thus $\Omega_d^{\text{Pin}^-}(\text{B}^2\mathbb{Z}_3)_3^{\wedge}=0$. Since $\text{H}^*(\text{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_2$, we have $\Omega_d^{\text{Pin}^-}(\text{B}^2\mathbb{Z}_3)_2^{\wedge}=\Omega_d^{\text{Pin}^-}$.

Hence $\Omega_d^{\text{Pin}^-}(\text{B}^2\mathbb{Z}_3)=\Omega_d^{\text{Pin}^-}$.

i	$\Omega_i^{\text{Pin}^-}(\text{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	0
5	0

Theorem 51.

The bordism invariant of $\Omega_2^{\text{Pin}^-}(\text{B}^2\mathbb{Z}_3)$ is ABK.

i	$\text{TP}_i(\text{Pin}^-\times\text{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	0
5	0

Theorem 52.

The 2d topological term is ABK.

5.5 $\mathrm{B}G_a : \mathrm{BPSU}(2), \mathrm{BPSU}(3)$

5.5.1 $\Omega_d^{\mathrm{O}}(\mathrm{BPSU}(2))$

$$\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2) = \mathcal{A}_2 \oplus 2\Sigma^2\mathcal{A}_2 \oplus \Sigma^3\mathcal{A}_2 \oplus 4\Sigma^4\mathcal{A}_2 \oplus 3\Sigma^5\mathcal{A}_2 \oplus \cdots . \tag{5.54}$$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O} \wedge (\mathrm{BPSU}(2))_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{BPSU}(2))_2^\wedge \tag{5.55}$$

i	$\Omega_i^{\mathrm{O}}(\mathrm{BPSU}(2))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^3$

Theorem 53.

The bordism invariants of $\Omega_2^{\mathrm{O}}(\mathrm{BPSU}(2))$ are w_2', w_1^2.

The bordism invariant of $\Omega_3^{\mathrm{O}}(\mathrm{BPSU}(2))$ is $w_3' = w_1 w_2'$.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{BPSU}(2))$ are $w_2'^2, w_1^4, w_1^2 w_2', w_2^2$.

The bordism invariants of $\Omega_5^{\mathrm{O}}(\mathrm{BPSU}(2))$ are $w_2 w_3, w_1^2 w_3', w_2' w_3'$.

i	$\mathrm{TP}_i(\mathrm{O} \times \mathrm{PSU}(2))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^3$

Theorem 54.

The 2d topological terms are w_2', w_1^2.

The 3d topological term is $w_3' = w_1 w_2'$.

The 4d topological terms are $w_2'^2, w_1^4, w_1^2 w_2', w_2^2$.

The 5d topological terms are $w_2 w_3, w_1^2 w_3', w_2' w_3'$.

5.5.2 $\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(2))$

$$\operatorname{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{BPSU}(2))_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{BPSU}(2))_2^{\wedge}. \tag{5.56}$$

The E_2 page is shown in Figure 36.

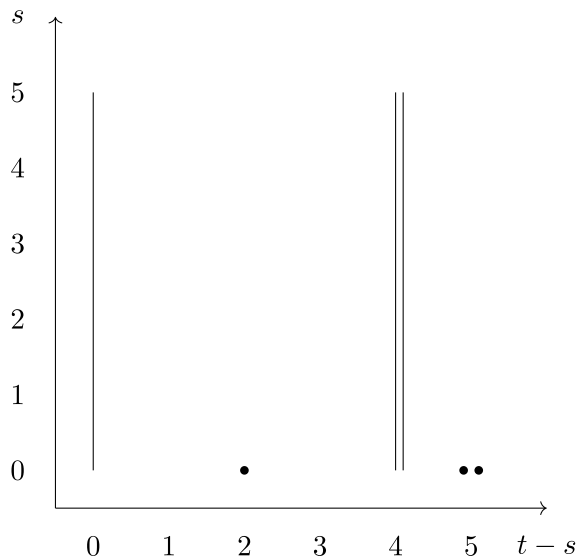


Figure 36: $\Omega_*^{\mathrm{SO}}(\mathrm{BPSU}(2))_2^{\wedge}$

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BPSU}(2))$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}^2$
5	$\mathbb{Z}_2^2$

Theorem 55.

The bordism invariant of $\Omega_2^{\mathrm{SO}}(\mathrm{BPSU}(2))$ is w_2'.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{BPSU}(2))$ are σ, p_1'.

The bordism invariants of $\Omega_5^{\mathrm{SO}}(\mathrm{BPSU}(2))$ are $w_2w_3, w_2'w_3'$.

The manifold generators of $\Omega_4^{\mathrm{SO}}(\mathrm{BPSU}(2))$ are $(\mathbb{CP}^2, 3)$ and $(\mathbb{CP}^2, L_{\mathbb{C}}+1)$ where n is the trivial real n-plane bundle and $L_{\mathbb{C}}$ is the tautological complex line bundle over $\mathbb{CP}^2$. Note that the principal SO(3)-bundle P associated to $L_{\mathbb{C}}+1$ is the induce bundle $P'\times_{\mathrm{SO}(2)}\mathrm{SO}(3)$ from P'

$$\begin{array}{ccc} S^1=\mathrm{SO}(2) & \longrightarrow & S^5 \\ & & \downarrow \\ & & \mathbb{CP}^2 \end{array} \tag{5.57}$$

by the group homomorphism ϕ : SO(2) $\to$ SO(3) which is the inclusion map, that means $P = P' \times_{\mathrm{SO}(2)} \mathrm{SO}(3) = (P' \times \mathrm{SO}(3))/\mathrm{SO}(2)$ which is the quotient of $P' \times \mathrm{SO}(3)$ by the right SO(2) action

$$(p, g)h = (ph, \phi(h^{-1})g). \tag{5.58}$$

$$p_1(L_{\mathbb{C}} + 1) = p_1(L_{\mathbb{C}}) = -c_2(L_{\mathbb{C}} \otimes_{\mathbb{R}} \mathbb{C}) = -c_2(L_{\mathbb{C}} \oplus \bar{L}_{\mathbb{C}}) = -c_1(L_{\mathbb{C}})c_1(\bar{L}_{\mathbb{C}}) = c_1(L_{\mathbb{C}})^2. \tag{5.59}$$

So

$$\int_{\mathbb{CP}^2} p_1(L_{\mathbb{C}} + 1) = 1. \tag{5.60}$$

i	$\mathrm{TP}_i(\mathrm{SO} \times \mathrm{PSU}(2))$
0	0
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}^2$
4	0
5	$\mathbb{Z}_2^2$

Theorem 56.

The 2d topological term is w_2'.

Since $p_1' = \mathrm{dCS}_3^{(\mathrm{SO}(3))}$, the 3d topological terms are $\frac{1}{3}\mathrm{CS}_3^{(TM)}$ and $\mathrm{CS}_3^{(\mathrm{SO}(3))}$.

The 5d topological terms are $w_2 w_3, w_2' w_3'$.

5.5.3 $\Omega_d^{\mathrm{Spin}}(\mathrm{BPSU}(2))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin} \wedge (\mathrm{BPSU}(2))_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BPSU}(2))_2^\wedge. \tag{5.61}$$

For $t - s < 8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BPSU}(2))_2^\wedge. \tag{5.62}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2)$ and the E_2 page are shown in Figure 37, 38.

Theorem 57.

The bordism invariants of $\Omega_2^{\mathrm{Spin}}(\mathrm{BPSU}(2))$ are w_2' and Arf.

By Wu formula (2.52), $w_2'^2 = \mathrm{Sq}^2(w_2') = (w_2(TM) + w_1(TM)^2)w_2' = 0$ on Spin 4-manifolds, $p_1' = w_2'^2 = 0 \mod 2$ on Spin 4-manifolds.

The bordism invariants of $\Omega_4^{\mathrm{Spin}}(\mathrm{BPSU}(2))$ are $\frac{\sigma}{16}$ and $\frac{p_1'}{2}$.

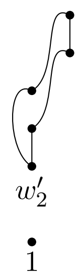


Figure 37: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2)$

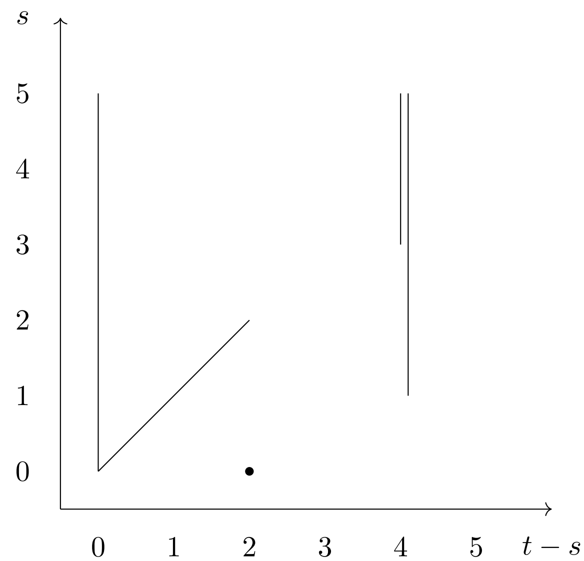


Figure 38: $\Omega_*^{\mathrm{Spin}}(\mathrm{BPSU}(2))_2^\wedge$

i	$\Omega_i^{\mathrm{Spin}}(\mathrm{BPSU}(2))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	0
4	$\mathbb{Z}^2$
5	0

i	$\mathrm{TP}_i(\mathrm{Spin} \times \mathrm{PSU}(2))$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}^2$
4	0
5	0

Theorem 58.

The 2d topological terms are w_2' and Arf.

The 3d topological terms are $\frac{1}{48}\mathrm{CS}_3^{(TM)}$ and $\frac{1}{2}\mathrm{CS}_3^{(\mathrm{SO}(3))}$.

5.5.4 $\Omega_d^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(MT\mathrm{Pin}^+ \wedge (\mathrm{BPSU}(2))_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))_2^\wedge. \tag{5.63}$$

$MT\mathrm{Pin}^+ = M\mathrm{Spin} \wedge S^1 \wedge MT\mathrm{O}(1)$.

For $t - s < 8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*-1}(MT\mathrm{O}(1), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))_2^\wedge. \tag{5.64}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2)$ and the E_2 page are shown in Figure 39, 40.

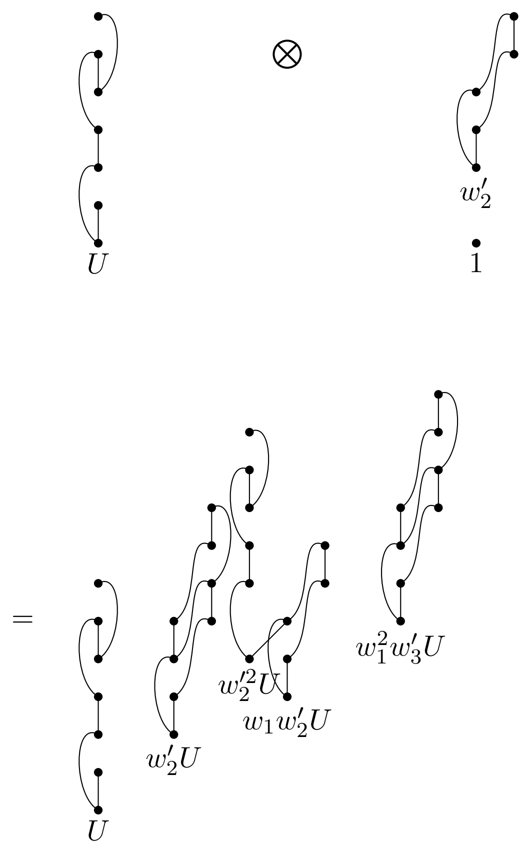


Figure 39: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{BPSU}(2), \mathbb{Z}_2)$

Theorem 59.

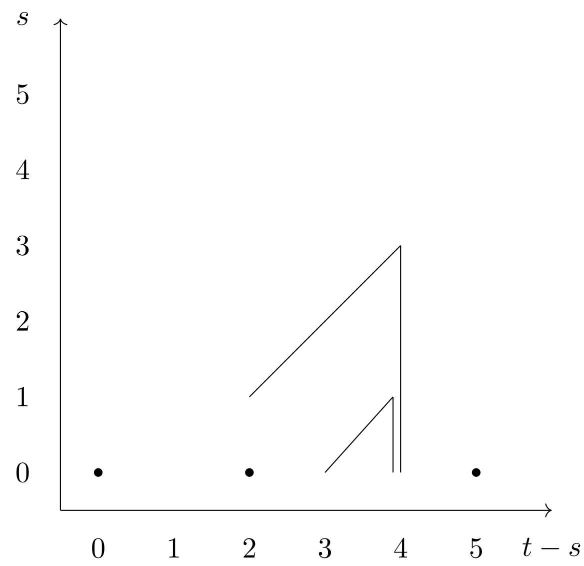


Figure 40: $\Omega_*^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))_2^\wedge$

i	$\Omega_i^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$
5	$\mathbb{Z}_2$

The bordism invariants of $\Omega_2^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))$ are w_2' and $w_1\tilde{\eta}$.

The bordism invariants of $\Omega_3^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))$ are $w_1w_2' = w_3'$ and $w_1\mathrm{Arf}$.

The bordism invariants of $\Omega_4^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))$ are $q_s(w_2')$ (this invariant has another form, see the footnotes of Table 3) and η.

The bordism invariant of $\Omega_5^{\mathrm{Pin}^+}(\mathrm{BPSU}(2))$ is $w_1^2w_3'(= w_2'w_3')$.

i	$\mathrm{TP}_i(\mathrm{Pin}^+ \times \mathrm{PSU}(2))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$
5	$\mathbb{Z}_2$

Theorem 60.

The 2d topological terms are w_2' and $w_1\tilde{\eta}$.

The 3d topological terms are $w_1w_2' = w_3'$ and $w_1\mathrm{Arf}$.

The 4d topological terms are $q_s(w_2')$ and η.

The 5d topological term is $w_1^2w_3'(=w_2'w_3')$.

5.5.5 $\Omega_d^{\text{Pin}^-}(\text{BPSU}(2))$

$$\text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(MT\text{Pin}^-\wedge(\text{BPSU}(2))_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Pin}^-}(\text{BPSU}(2))_2^\wedge. \tag{5.65}$$

$MT\text{Pin}^-=M\text{Spin}\wedge S^{-1}\wedge M\text{O}(1)$.

For $t-s<8$,

$$\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(\text{H}^{*+1}(M\text{O}(1),\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(2),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Pin}^-}(\text{BPSU}(2))_2^\wedge. \tag{5.66}$$

The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*+1}(M\text{O}(1),\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(2),\mathbb{Z}_2)$ and the E_2 page are shown in Figure 41, 42.

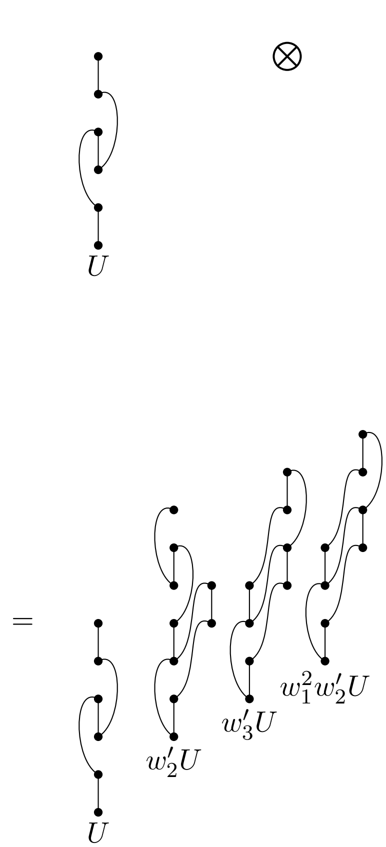


Figure 41: The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*+1}(M\text{O}(1),\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(2),\mathbb{Z}_2)$

Theorem 61.

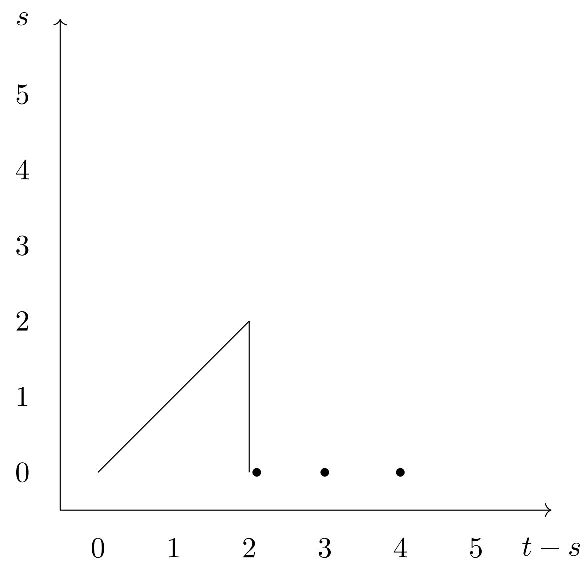


Figure 42: $\Omega_*^{\mathrm{Pin}^-}(\mathrm{BPSU}(2))_2^\wedge$

i	$\Omega_i^{\mathrm{Pin}^-}(\mathrm{BPSU}(2))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2 \times \mathbb{Z}_8$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2$
5	0

The bordism invariants of $\Omega_2^{\mathrm{Pin}^-}(\mathrm{BPSU}(2))$ are w_2' and ABK.

The bordism invariant of $\Omega_3^{\mathrm{Pin}^-}(\mathrm{BPSU}(2))$ is $w_1w_2' = w_3'$.

The bordism invariant of $\Omega_4^{\mathrm{Pin}^-}(\mathrm{BPSU}(2))$ is $w_1^2w_2'$.

i	$\mathrm{TP}_i(\mathrm{Pin}^- \times \mathrm{PSU}(2))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2 \times \mathbb{Z}_8$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2$
5	0

Theorem 62.

The 2d topological terms are w_2' and ABK.

The 3d topological term is $w_1w_2' = w_3'$.

The 4d topological term is $w_1^2w_2'$.

5.5.6 $\Omega_d^{\mathrm{O}}(\mathrm{BPSU}(3))$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_3)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{BPSU}(3))_3^{\wedge}. \tag{5.67}$$

Since $\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_3)=0$, $\Omega_d^{\mathrm{O}}(\mathrm{BPSU}(3))_3^{\wedge}=0$.

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{BPSU}(3))_2^{\wedge}. \tag{5.68}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2)=\mathcal{A}_2\oplus\Sigma^2\mathcal{A}_2\oplus3\Sigma^4\mathcal{A}_2\oplus\Sigma^5\mathcal{A}_2\oplus\cdots. \tag{5.69}$$

i	$\Omega_i^{\mathrm{O}}(\mathrm{BPSU}(3))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2$

Theorem 63.

The bordism invariant of $\Omega_2^{\mathrm{O}}(\mathrm{BPSU}(3))$ is w_1^2.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{BPSU}(3))$ are $w_1^4, w_2^2, c_2(\mod 2)$.

The bordism invariant of $\Omega_5^{\mathrm{O}}(\mathrm{BPSU}(3))$ is w_2w_3.

i	$\mathrm{TP}_i(\mathrm{O}\times\mathrm{PSU}(3))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2$

Theorem 64.

The 2d topological term is w_1^2.

The 4d topological terms are $w_1^4, w_2^2, c_2(\mod 2)$.

The 5d topological term is w_2w_3.

5.5.7 $\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(3))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{BPSU}(3))_2^{\wedge}. \tag{5.70}$$

$$\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2)=\mathbb{Z}_2[c_2,c_3]. \tag{5.71}$$

The E_2 page is shown in Figure 43.

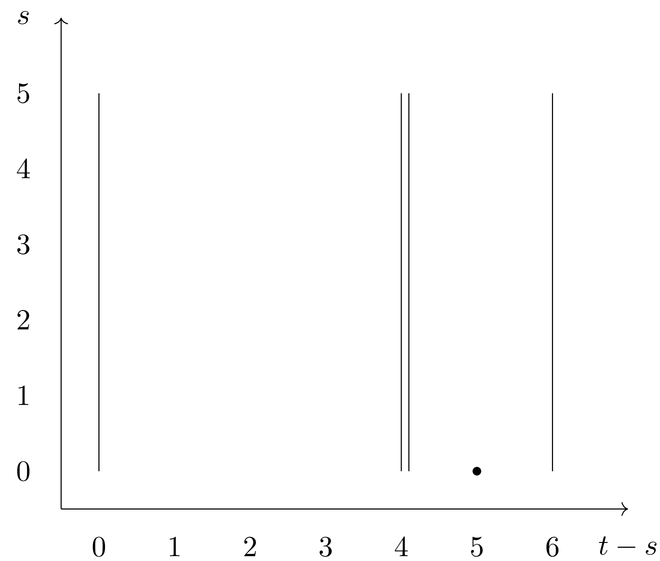


Figure 43: $\Omega_*^{\mathrm{SO}}(\mathrm{BPSU}(3))_2^\wedge$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_3)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{BPSU}(3))_3^\wedge. \tag{5.72}$$

$$\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_3)=(\mathbb{Z}_3[z_2,z_8,z_{12}]\otimes\Lambda_{\mathbb{Z}_3}(z_3,z_7))/(z_2z_3,z_2z_7,z_2z_8+z_3z_7) \tag{5.73}$$

$\beta_{(3,3)}z_2=z_3$, $\beta_{(3,3)}z_2^2=2z_2z_3=0$, $\beta_{(3,3)}z_2^3=0$.

The E_2 page is shown in Figure 44.

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BPSU}(3))$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_3$
3	0
4	$\mathbb{Z}^2$
5	$\mathbb{Z}_2$
6	$\mathbb{Z}$

Theorem 65.

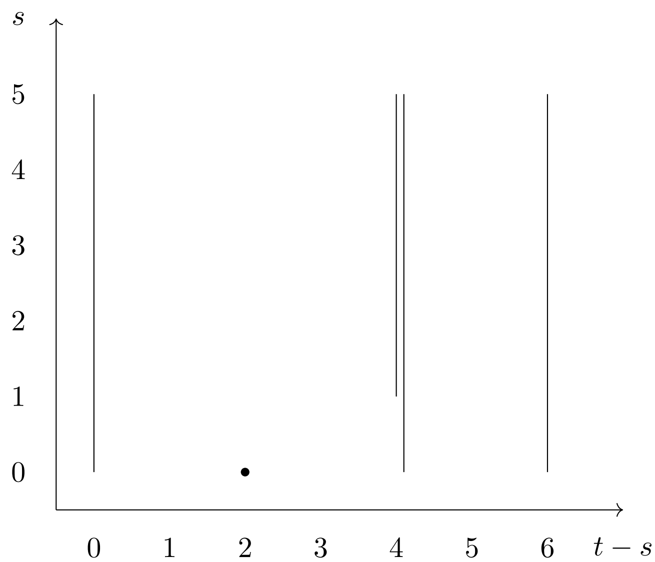


Figure 44: $\Omega_*^{\mathrm{SO}}(\mathrm{BPSU}(3))_3^\wedge$

The bordism invariant of $\Omega_2^{\mathrm{SO}}(\mathrm{BPSU}(3))$ is z_2.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{BPSU}(3))$ are σ, c_2.

The bordism invariant of $\Omega_5^{\mathrm{SO}}(\mathrm{BPSU}(3))$ is w_2w_3.

The bordism invariant of $\Omega_6^{\mathrm{SO}}(\mathrm{BPSU}(3))$ is c_3.

The manifold generators of $\Omega_4^{\mathrm{SO}}(\mathrm{BPSU}(3))$ are $(\mathbb{CP}^2, \mathbb{CP}^2 \times \mathrm{PSU}(3))$ and (S^4, H) where H is the induced bundle from the Hopf fibration H'

$$\begin{array}{ccc} S^3 = \mathrm{SU}(2) & \longrightarrow & S^7 \\ & & \downarrow \\ & & S^4 \end{array} \tag{5.74}$$

by the group homomorphism $\rho : \mathrm{SU}(2) \to \mathrm{PSU}(3)$ which is the composition of the inclusion map $\mathrm{SU}(2) \to \mathrm{SU}(3)$ and the quotient map $\mathrm{SU}(3) \to \mathrm{PSU}(3)$, that means $H = H' \times_{\mathrm{SU}(2)} \mathrm{PSU}(3) = (H' \times \mathrm{PSU}(3))/\mathrm{SU}(2)$ which is the quotient of $H' \times \mathrm{PSU}(3)$ by the right $\mathrm{SU}(2)$ action

$$(p, g)h = (ph, \rho(h^{-1})g). \tag{5.75}$$

Theorem 66.

The 2d topological term is z_2.

Since $c_2 = \mathrm{dCS}_3^{(\mathrm{PSU}(3))}$, the 3d topological terms are $\frac{1}{3}\mathrm{CS}_3^{(TM)}$ and $\mathrm{CS}_3^{(\mathrm{PSU}(3))}$.

Since $c_3 = \mathrm{dCS}_5^{(\mathrm{PSU}(3))}$, the 5d topological term are $\mathrm{CS}_5^{(\mathrm{PSU}(3))}$ and w_2w_3.

i	$\mathrm{TP}_i(\mathrm{SO}\times\mathrm{PSU}(3))$
0	0
1	0
2	$\mathbb{Z}_3$
3	$\mathbb{Z}^2$
4	0
5	$\mathbb{Z}\times\mathbb{Z}_2$

5.5.8 $\Omega_d^{\mathrm{Spin}}(\mathrm{BPSU}(3))$

For $t-s<8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BPSU}(3))_2^\wedge. \tag{5.76}$$

The E_2 page is shown in Figure 45.

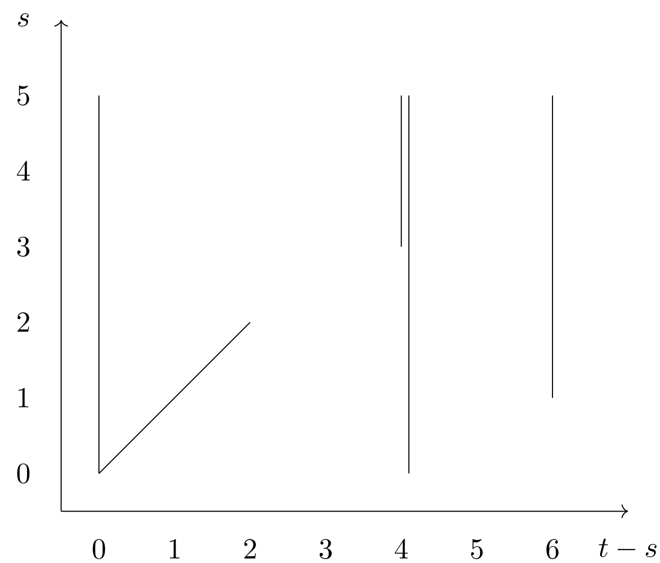


Figure 45: $\Omega_*^{\mathrm{Spin}}(\mathrm{BPSU}(3))_2^\wedge$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_3)\otimes\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BPSU}(3))_3^\wedge. \tag{5.77}$$

Since $\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_3)=\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_3)$, the E_2 page is shown in Figure 46.

Theorem 67.

The bordism invariants of $\Omega_2^{\mathrm{Spin}}(\mathrm{BPSU}(3))$ are Arf and z_2.

The bordism invariants of $\Omega_4^{\mathrm{Spin}}(\mathrm{BPSU}(3))$ are $\frac{\sigma}{16}$ and c_2.

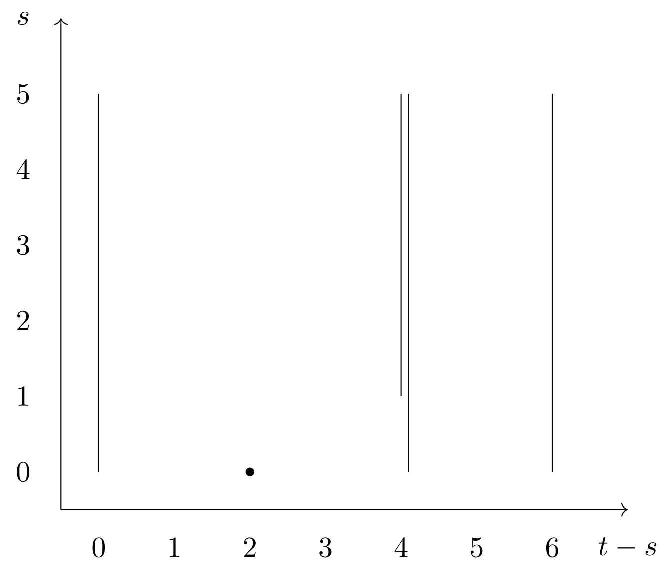


Figure 46: $\Omega_*^{\text{Spin}}(\text{BPSU}(3))_3^\wedge$

i	$\Omega_i^{\text{Spin}}(\text{BPSU}(3))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2 \times \mathbb{Z}_3$
3	0
4	$\mathbb{Z}^2$
5	0
6	$\mathbb{Z}$

By Wu formula (2.52), $c_3 = \text{Sq}^2 c_2 = (w_2(TM) + w_1^2(TM))c_2 = 0 \mod 2$ on Spin 6-manifolds.

The bordism invariant of $\Omega_6^{\text{Spin}}(\text{BPSU}(3))$ is $\frac{c_3}{2}$.

i	$\text{TP}_i(\text{Spin} \times \text{PSU}(3))$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2 \times \mathbb{Z}_3$
3	$\mathbb{Z}^2$
4	0
5	$\mathbb{Z}$

Theorem 68.

The 2d topological terms are Arf and z_2.

The 3d topological terms are $\frac{1}{48}\text{CS}_3^{(TM)}$ and $\text{CS}_3^{(\text{PSU}(3))}$.

The 5d topological term is $\frac{1}{2}\text{CS}_5^{(\text{PSU}(3))}$.

5.5.9 $\Omega_d^{\text{Pin}^+}(\text{BPSU}(3))$

$$\text{Ext}_{\mathcal{A}_3}^{s,t}(\text{H}^*(M\text{Pin}^-,\mathbb{Z}_3)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\text{Pin}^+}(\text{BPSU}(3))_3^{\wedge}. \tag{5.78}$$

Since $\text{H}^*(M\text{Pin}^-,\mathbb{Z}_3)=\text{H}^*(MO,\mathbb{Z}_3)=0$, $\Omega_{t-s}^{\text{Pin}^+}(\text{BPSU}(3))_3^{\wedge}=0$.

$$\text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(M\text{Pin}^-,\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Pin}^+}(\text{BPSU}(3))_2^{\wedge}. \tag{5.79}$$

For $t-s<8$,

$$\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(\text{H}^{*-1}(MT\text{O}(1),\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Pin}^+}(\text{BPSU}(3))_2^{\wedge}. \tag{5.80}$$

The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*-1}(MT\text{O}(1),\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2)$ and the E_2 page are shown in Figure 47, 48.

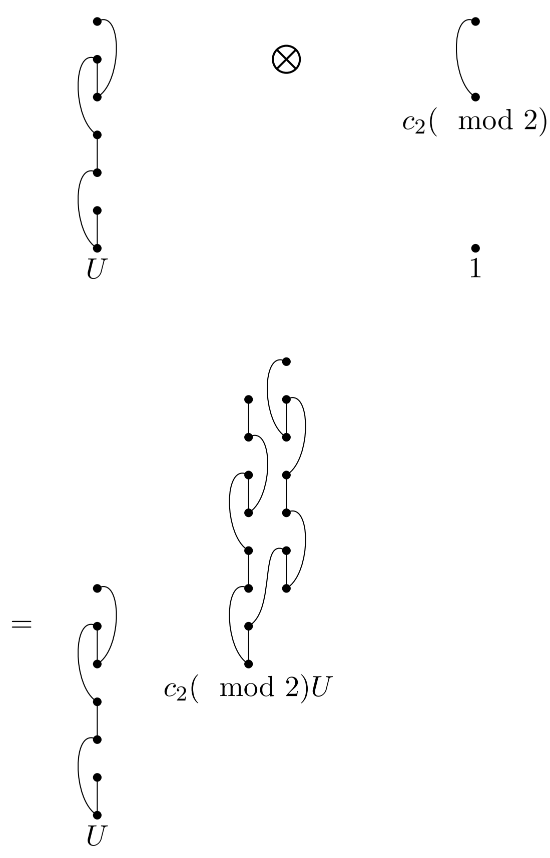


Figure 47: The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*-1}(MT\text{O}(1),\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2)$

Theorem 69.

The bordism invariant of $\Omega_2^{\text{Pin}^+}(\text{BPSU}(3))$ is $w_1\tilde{\eta}$.

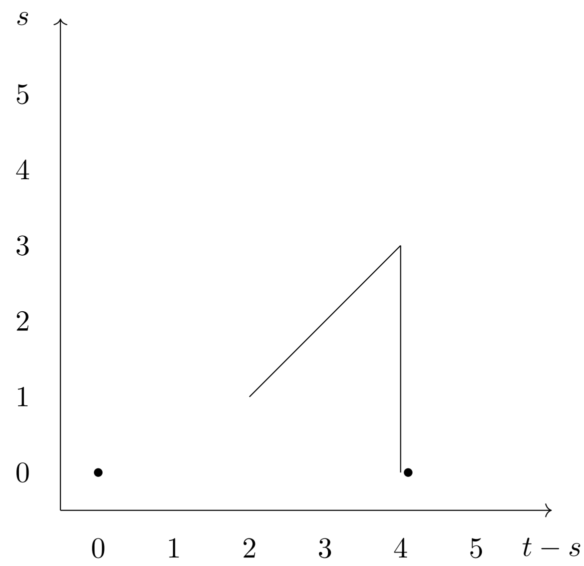


Figure 48: $\Omega_*^{\mathrm{Pin}^+}(\mathrm{BPSU}(3))_2^\wedge$

i	$\Omega_i^{\mathrm{Pin}^+}(\mathrm{BPSU}(3))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$
5	0

The bordism invariant of $\Omega_3^{\mathrm{Pin}^+}(\mathrm{BPSU}(3))$ is $w_1\mathrm{Arf}$.

The bordism invariants of $\Omega_4^{\mathrm{Pin}^+}(\mathrm{BPSU}(3))$ are $c_2(\mod 2)$ and η.

i	$\mathrm{TP}_i(\mathrm{Pin}^+ \times \mathrm{PSU}(3))$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$
5	0

Theorem 70.

The 2d topological term is $w_1\tilde{\eta}$.

The 3d topological term is $w_1\mathrm{Arf}$.

The 4d topological terms are $c_2(\mod 2)$ and η.

5.5.10 $\Omega_d^{\text{Pin}^-}(\text{BPSU}(3))$

$$\text{Ext}_{\mathcal{A}_3}^{s,t}(\text{H}^*(M\text{Pin}^+,\mathbb{Z}_3)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\text{Pin}^-}(\text{BPSU}(3))_3^\wedge. \tag{5.81}$$

Since $\text{H}^*(M\text{Pin}^+,\mathbb{Z}_3)=\text{H}^*(M\text{O},\mathbb{Z}_3)=0$, $\Omega_{t-s}^{\text{Pin}^-}(\text{BPSU}(3))_3^\wedge=0$.

$$\text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(M\text{Pin}^+,\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Pin}^-}(\text{BPSU}(3))_2^\wedge. \tag{5.82}$$

For $t-s<8$,

$$\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(\text{H}^{*+1}(M\text{O}(1),\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Pin}^-}(\text{BPSU}(3))_2^\wedge. \tag{5.83}$$

The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*+1}(M\text{O}(1),\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2)$ and the E_2 page are shown in Figure 49, 50.

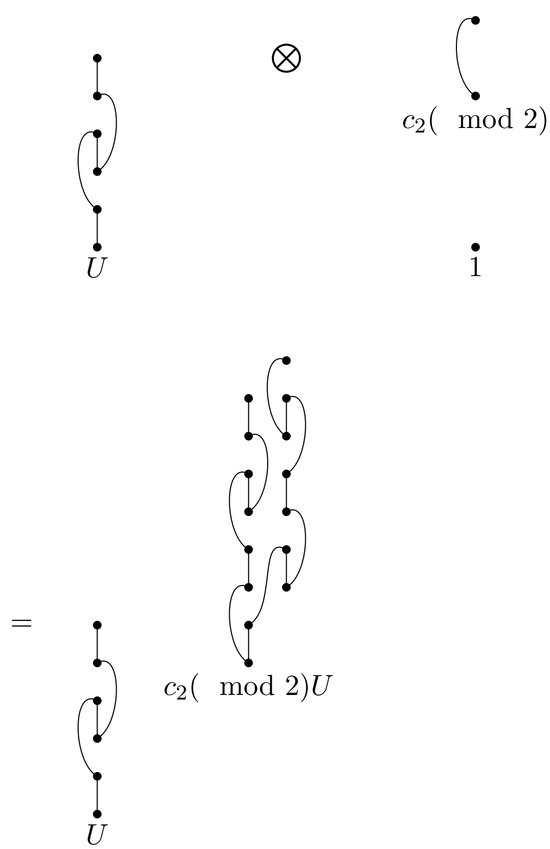


Figure 49: The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*+1}(M\text{O}(1),\mathbb{Z}_2)\otimes\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2)$

Theorem 71.

The bordism invariant of $\Omega_2^{\text{Pin}^-}(\text{BPSU}(3))$ is ABK.

The bordism invariant of $\Omega_4^{\text{Pin}^-}(\text{BPSU}(3))$ is $c_2(\mod 2)$.

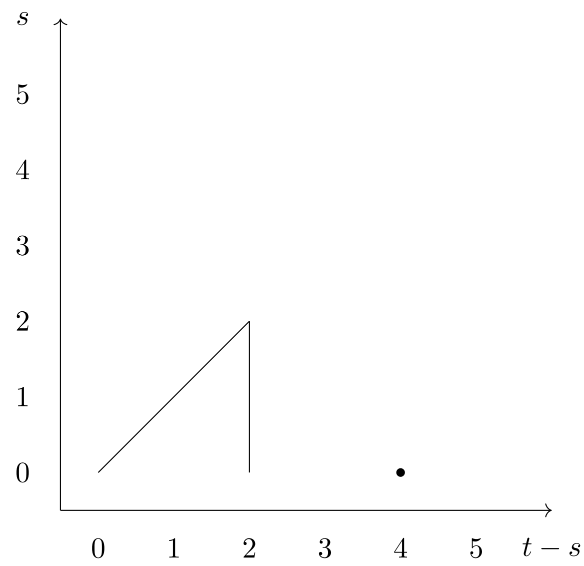


Figure 50: $\Omega_*^{\mathrm{Pin}^-}(\mathrm{BPSU}(3))_2^\wedge$

i	$\Omega_i^{\mathrm{Pin}^-}(\mathrm{BPSU}(3))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	$\mathbb{Z}_2$
5	0

i	$\mathrm{TP}_i(\mathrm{Pin}^- \times \mathrm{PSU}(3))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	$\mathbb{Z}_2$
5	0

Theorem 72.

The 2d topological term is ABK.

The 4d topological term is $c_2(\mod 2)$.

5.6 $(\mathrm{B}G_a, \mathrm{B}^2 G_b) : (\mathrm{B}\mathbb{Z}_2, \mathrm{B}^2\mathbb{Z}_2), (\mathrm{B}\mathbb{Z}_3, \mathrm{B}^2\mathbb{Z}_3)$

5.6.1 $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O} \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \\ \Rightarrow\ & \pi_{t-s}(M\mathrm{O} \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2). \end{aligned} \tag{5.84}$$

$$\begin{aligned} &\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \otimes \mathbb{Z}_2[a, x_2, x_3, x_5, x_9, \dots] \\ =\ & \mathcal{A}_2 \oplus \Sigma\mathcal{A}_2 \oplus 3\Sigma^2\mathcal{A}_2 \oplus 4\Sigma^3\mathcal{A}_2 \oplus 8\Sigma^4\mathcal{A}_2 \oplus 12\Sigma^5\mathcal{A}_2 \oplus \cdots \end{aligned} \tag{5.85}$$

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^8$
5	$\mathbb{Z}_2^{12}$

Theorem 73.

The bordism invariants of $\Omega_2^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are a^2, x_2, w_1^2.

The bordism invariants of $\Omega_3^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $x_3 = w_1 x_2, a x_2, a w_1^2, a^3$.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $w_1^4, w_2^2, a^4, a^2 x_2, a x_3, x_2^2, w_1^2 a^2, w_1^2 x_2$.

The bordism invariants of $\Omega_5^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are

$$a^5, a^2 x_3, a^3 x_2, a^3 w_1^2, a x_2^2, a w_1^4, a x_2 w_1^2, a w_2^2, x_2 x_3, w_1^2 x_3, x_5, w_2 w_3.$$

i	$\mathrm{TP}_i(\mathrm{O} \times \mathbb{Z}_2 \times \mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^8$
5	$\mathbb{Z}_2^{12}$

Theorem 74.

The 2d topological terms are a^2, x_2, w_1^2.

The 3d topological terms are $x_3 = w_1 x_2, a x_2, a w_1^2, a^3$.

The 4d topological terms are $w_1^4, w_2^2, a^4, a^2x_2, ax_3, x_2^2, w_1^2a^2, w_1^2x_2$.

The 5d topological terms are

$$a^5, a^2x_3, a^3x_2, a^3w_1^2, ax_2^2, aw_1^4, ax_2w_1^2, aw_2^2, x_2x_3, w_1^2x_3, x_5, w_2w_3.$$

5.6.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$\begin{aligned} & E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO} \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \\ \Rightarrow \quad & \pi_{t-s}(M\mathrm{SO} \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2). \end{aligned} \tag{5.86}$$

There is a differential d_2 corresponding to the Bockstein homomorphism $\beta_{(2,4)} : \mathrm{H}^*(-, \mathbb{Z}_4) \to \mathrm{H}^{*+1}(-, \mathbb{Z}_2)$ associated to $0 \to \mathbb{Z}_2 \to \mathbb{Z}_8 \to \mathbb{Z}_4 \to 0$ [57]. See 2.5 for the definition of Bockstein homomorphisms.

By (5.32), there is a differential such that $d_2(x_2x_3 + x_5) = x_2^2h_0^2$.

The E_2 page is shown in Figure 51.

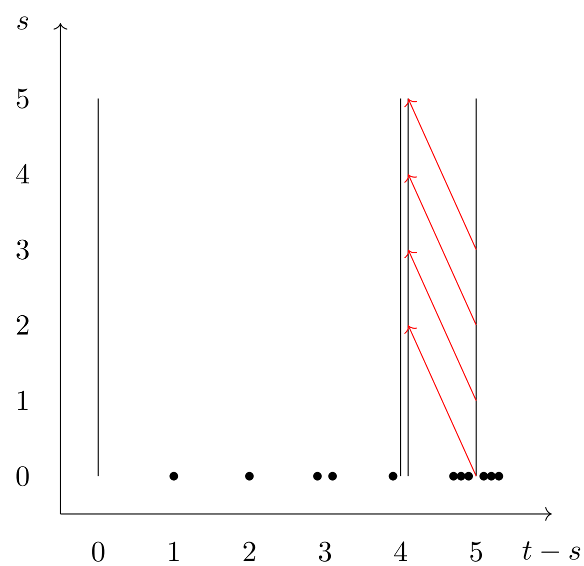


Figure 51: $\Omega_*^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

Theorem 75.

The bordism invariant of $\Omega_2^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ is x_2.

The bordism invariants of $\Omega_3^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are ax_2, a^3.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are σ, $ax_3(= a^2x_2)$ and $\mathcal{P}_2(x_2)$.

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$
5	$\mathbb{Z}_2^6$

The bordism invariants of $\Omega_5^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $ax_2^2, a^5, x_5, a^3x_2, w_2w_3, aw_2^2$.

i	$\mathrm{TP}_i(\mathrm{SO} \times \mathbb{Z}_2 \times \mathrm{B}\mathbb{Z}_2)$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z} \times \mathbb{Z}_2^2$
4	$\mathbb{Z}_2 \times \mathbb{Z}_4$
5	$\mathbb{Z}_2^6$

Theorem 76.

The 2d topological term is x_2.

The 3d topological terms are $\frac{1}{3}\mathrm{CS}_3^{(TM)}, ax_2, a^3$.

The 4d topological terms are $ax_3 (= a^2x_2)$ and $\mathcal{P}_2(x_2)$.

The 5d topological terms are $ax_2^2, a^5, x_5, a^3x_2, w_2w_3, aw_2^2$.

5.6.3 $\Omega_d^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin} \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \\ \Rightarrow\ & \pi_{t-s}(M\mathrm{Spin} \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2). \end{aligned} \tag{5.87}$$

For $t - s < 8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2). \tag{5.88}$$

$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[a, x_2, x_3, x_5, x_9, \dots]$ where $\mathrm{Sq}^1 x_2 = x_3$, $\mathrm{Sq}^2 x_2 = x_2^2$, $\mathrm{Sq}^1 x_3 = 0$, $\mathrm{Sq}^2 x_3 = x_5$, $\mathrm{Sq}^1 x_5 = \mathrm{Sq}^2 x_2^2 = x_3^2$, $\mathrm{Sq}^2 x_5 = 0$.

There is a differential d_2 corresponding to the Bockstein homomorphism $\beta_{(2,4)} : \mathrm{H}^*(-, \mathbb{Z}_4) \to \mathrm{H}^{*+1}(-, \mathbb{Z}_2)$ associated to $0 \to \mathbb{Z}_2 \to \mathbb{Z}_8 \to \mathbb{Z}_4 \to 0$ [57]. See 2.5 for the definition of Bockstein homomorphisms.

By (5.32), there is a differential such that $d_2(x_2x_3 + x_5) = x_2^2 h_0^2$.

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ and the E_2 page are shown in Figure 52, 53.

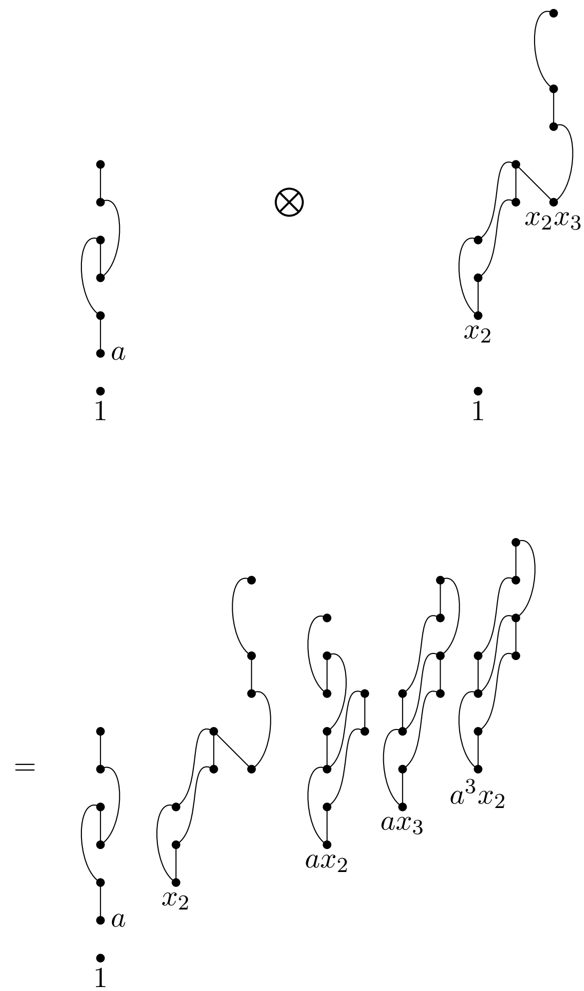


Figure 52: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2 \times \mathbb{Z}_8$
4	$\mathbb{Z} \times \mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 77.

The bordism invariants of $\Omega_2^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $x_2, \mathrm{Arf}, a\tilde{\eta}$.

The bordism invariants of $\Omega_3^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $ax_2, a\mathrm{ABK}$.

The bordism invariants of $\Omega_4^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $\frac{\sigma}{16}$, $ax_3(=a^2x_2)$ and $\frac{\mathcal{P}_2(x_2)}{2}$.

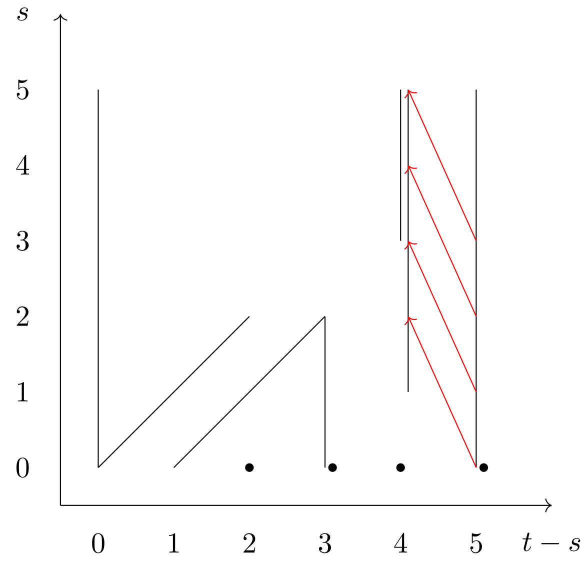


Figure 53: $\Omega_*^{\text{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

The bordism invariant of $\Omega_5^{\text{Spin}}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ is $a^3 x_2$.

i	$\mathrm{TP}_i(\mathrm{Spin} \times \mathbb{Z}_2 \times \mathrm{B}\mathbb{Z}_2)$
0	0
1	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_8$
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 78.

The 2d topological terms are $x_2, \mathrm{Arf}, a\tilde{\eta}$.

The 3d topological terms are $\frac{1}{48}\mathrm{CS}_3^{(TM)}, ax_2, a\mathrm{ABK}$.

The 4d topological terms are $ax_3(= a^2x_2)$ and $\frac{\mathcal{P}_2(x_2)}{2}$.

The 5d topological term is a^3x_2.

5.6.4 $\Omega_d^{\text{Pin}^+}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^- \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \\ \Rightarrow \quad & \pi_{t-s}(M\mathrm{Pin}^- \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\text{Pin}^+}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2). \end{aligned} \tag{5.89}$$

$M\mathrm{Pin}^- = MT\mathrm{Pin}^+ \sim M\mathrm{Spin} \wedge S^1 \wedge MT\mathrm{O}(1)$.

For $t-s<8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2). \quad (5.90)$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ and the E_2 page are shown in Figure 54, 55.

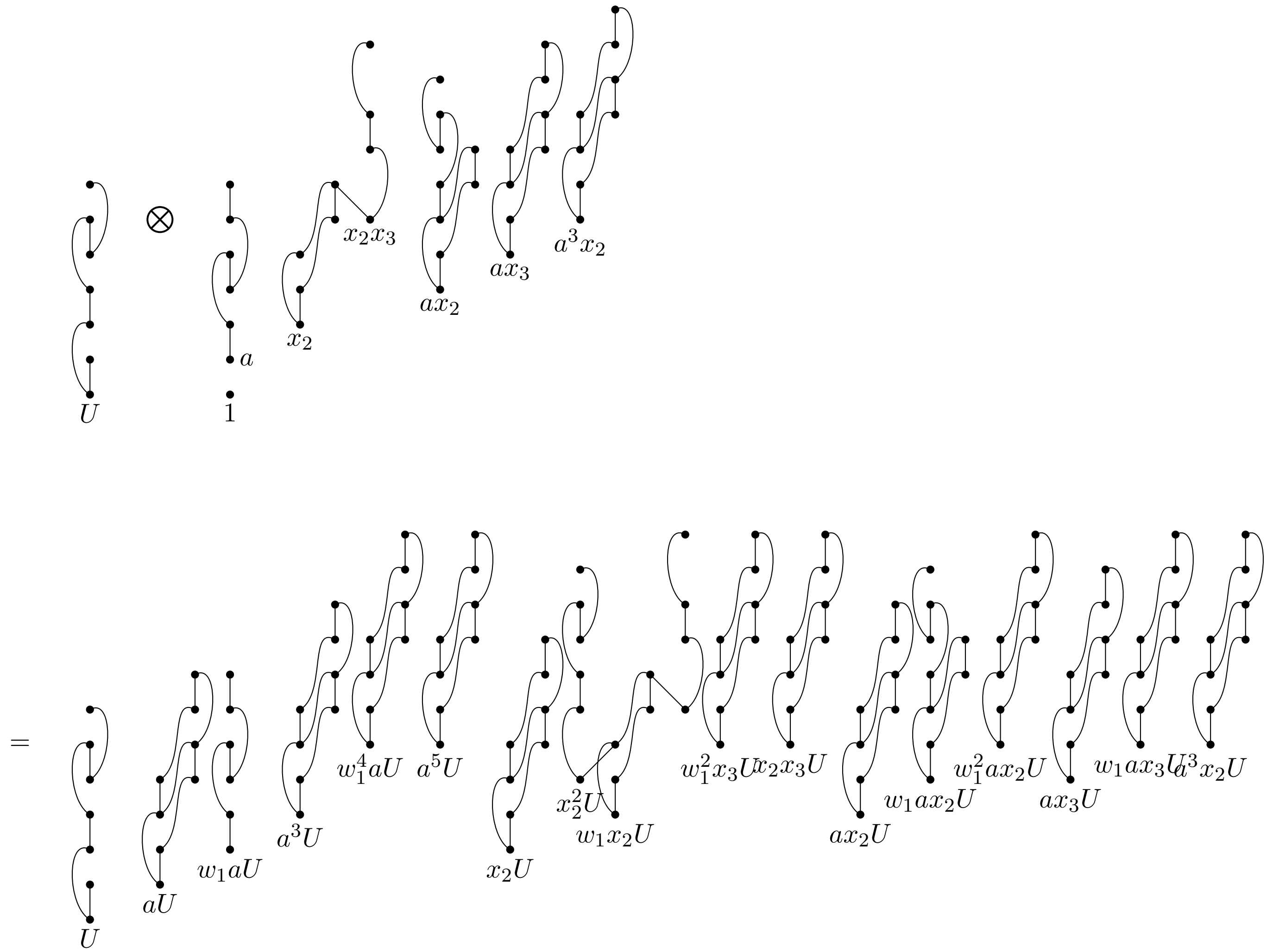


Figure 54: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^5$
4	$\mathbb{Z}_2^2\times\mathbb{Z}_4\times\mathbb{Z}_8\times\mathbb{Z}_{16}$
5	$\mathbb{Z}_2^7$

Theorem 79.

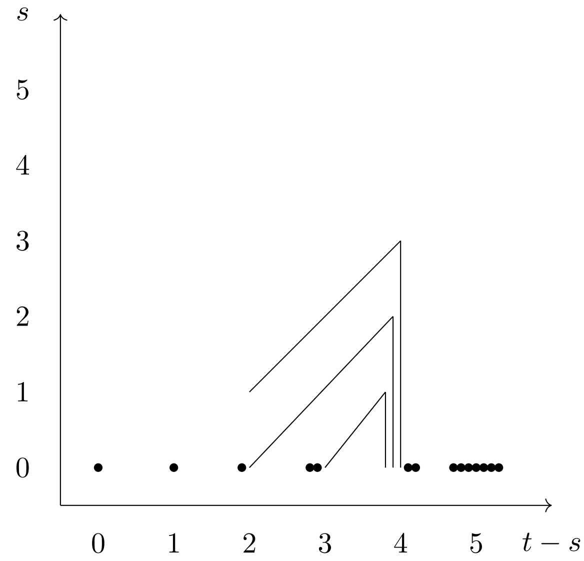


Figure 55: $\Omega_*^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

The bordism invariants of $\Omega_2^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $w_1 a = a^2, x_2, w_1\tilde{\eta}$.

The bordism invariants of $\Omega_3^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $a^3, w_1 x_2 = x_3, a x_2, w_1 a\tilde{\eta}, w_1\mathrm{Arf}$.

The bordism invariants of $\Omega_4^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are $a x_3, w_1 a x_2 (= a^2 x_2 + a x_3), q_s(x_2), w_1 a(\mathrm{ABK}), \eta$.

The bordism invariants of $\Omega_5^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$ are

$$w_1^4 a, a^5 (= w_1^2 a^3), w_1^2 x_3 (= x_5), x_2 x_3, w_1^2 a x_2 (= a x_2^2 + a^2 x_3), w_1 a x_3 (= a^2 x_3), a^3 x_2.$$

i	$\mathrm{TP}_i(\mathrm{Pin}^+ \times \mathbb{Z}_2 \times \mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^5$
4	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_{16}$
5	$\mathbb{Z}_2^7$

Theorem 80.

The 2d topological terms are $w_1 a = a^2, x_2, w_1\tilde{\eta}$.

The 3d topological terms are $a^3, w_1 x_2 = x_3, a x_2, w_1 a\tilde{\eta}, w_1\mathrm{Arf}$.

The 4d topological terms are $a x_3, w_1 a x_2 (= a^2 x_2 + a x_3), q_s(x_2), w_1 a(\mathrm{ABK}), \eta$.

The 5d topological terms are

$$w_1^4 a, a^5 (= w_1^2 a^3), w_1^2 x_3 (= x_5), x_2 x_3, w_1^2 a x_2 (= a x_2^2 + a^2 x_3), w_1 a x_3 (= a^2 x_3), a^3 x_2.$$

5.6.5 $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)$

Since the computation involves no odd torsion, we can use the Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+ \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+, \mathbb{Z}_2), \mathbb{Z}_2) \\ \Rightarrow \quad & \pi_{t-s}(M\mathrm{Pin}^+ \wedge (\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2)_+)_2^\wedge = \Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2). \end{aligned} \tag{5.91}$$

$M\mathrm{Pin}^+ = MT\mathrm{Pin}^- \sim M\mathrm{Spin} \wedge S^{-1} \wedge M\mathrm{O}(1)$.

For $t - s < 8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+1}(M\mathrm{O}(1), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2). \tag{5.92}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ and the E_2 page are shown in Figure 56, 57.

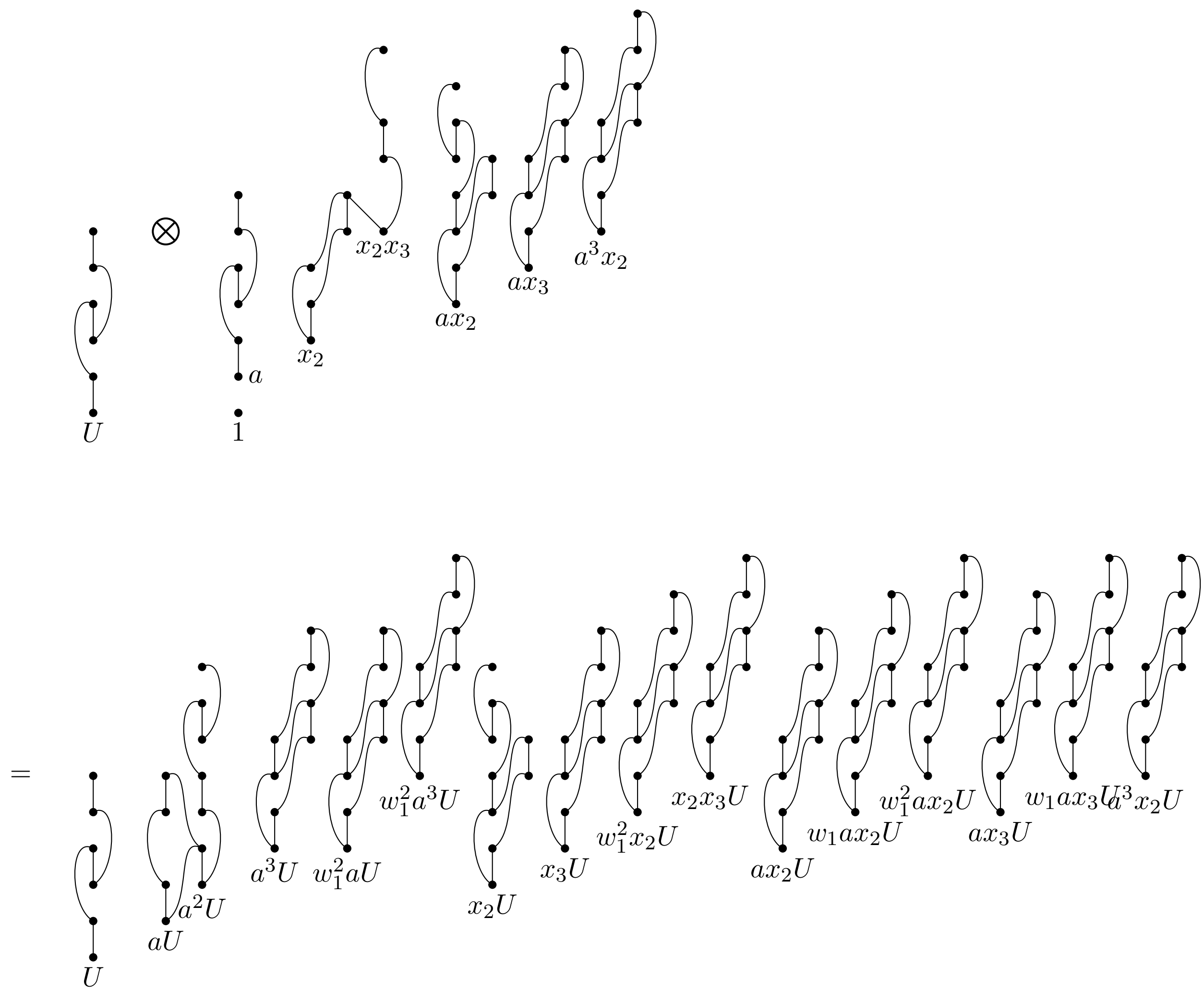


Figure 56: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$

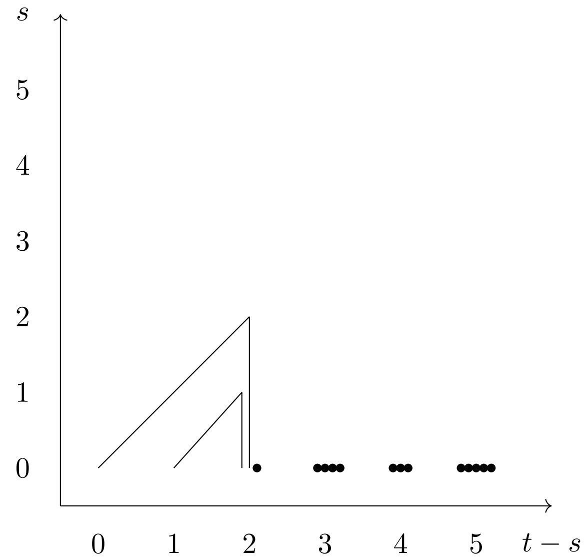


Figure 57: $\Omega_*^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_8$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2^5$

Theorem 81.

The bordism invariants of $\Omega_2^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2)$ are $x_2, q(a), \mathrm{ABK}$. ($q(a)$ is explained in the footnotes of Table 1.)

The bordism invariants of $\Omega_3^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2)$ are $a^3, w_1^2a, x_3 = w_1x_2, ax_2$.

The bordism invariants of $\Omega_4^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2)$ are $w_1^2x_2, w_1ax_2(= a^2x_2 + ax_3), ax_3$.

The bordism invariants of $\Omega_5^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_2\times\mathrm{B}^2\mathbb{Z}_2)$ are $w_1^2a^3, x_2x_3, w_1^2ax_2, w_1ax_3(= a^2x_3), a^3x_2$.

i	$\mathrm{TP}_i(\mathrm{Pin}^-\times\mathbb{Z}_2\times\mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_8$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2^5$

Theorem 82.

The 2d topological terms are $x_2, q(a), \text{ABK}$.

The 3d topological terms are $a^3, w_1^2 a, x_3 = w_1 x_2, a x_2$.

The 4d topological terms are $w_1^2 x_2, w_1 a x_2 (= a^2 x_2 + a x_3), a x_3$.

The 5d topological terms are $w_1^2 a^3, x_2 x_3, w_1^2 a x_2, w_1 a x_3 (= a^2 x_3), a^3 x_2$.

5.6.6 $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O} \wedge (\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{5.93}$$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{O} \wedge (\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{5.94}$$

Since $\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_3) = 0$, we have $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge = 0$.

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_2) = \mathbb{Z}_2$, we have $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge = \Omega_d^{\mathrm{O}}$.

Hence $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3) = \Omega_d^{\mathrm{O}}$.

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 83.

The bordism invariant of $\Omega_2^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ is w_1^2.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ are w_1^4, w_2^2.

The bordism invariant of $\Omega_5^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ is $w_2 w_3$.

i	$\mathrm{TP}_i(\mathrm{O} \times \mathbb{Z}_3 \times \mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 84.

The 2d topological term is w_1^2.

The 4d topological terms are w_1^4, w_2^2.

The 5d topological term is $w_2 w_3$.

5.6.7 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO} \wedge (\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{5.95}$$

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_2) = \mathbb{Z}_2$, we have $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge = \Omega_d^{\mathrm{SO}}$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO} \wedge (\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{5.96}$$

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3[b', x_2', x_8', \dots] \otimes \Lambda_{\mathbb{Z}_3}(a', x_3', x_7', \dots) \tag{5.97}$$

$\beta_{(3,3)} a' = b'$, $\beta_{(3,3)} x_2' = x_3'$, $\beta_{(3,3)} x_2'^2 = 2x_2'x_3'$.

The E_2 page is shown in Figure 58.

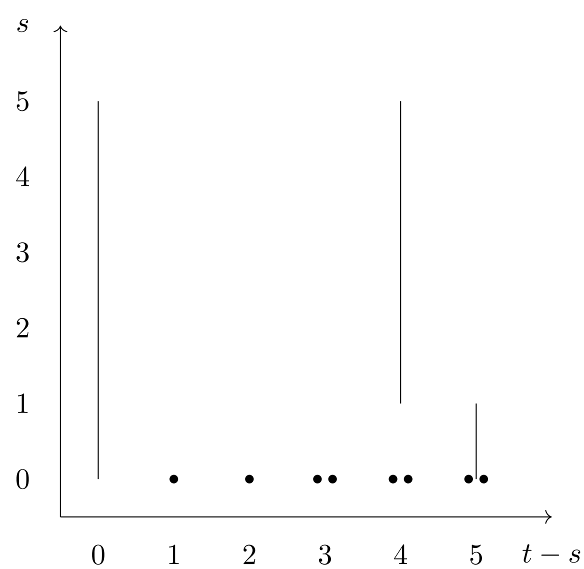


Figure 58: $\Omega_*^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge$

Hence we have the following

Theorem 85.

The bordism invariant of $\Omega_2^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ is x_2'.

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_3$
2	$\mathbb{Z}_3$
3	$\mathbb{Z}_3^2$
4	$\mathbb{Z}\times\mathbb{Z}_3^2$
5	$\mathbb{Z}_2\times\mathbb{Z}_3^2\times\mathbb{Z}_9$

The bordism invariants of $\Omega_3^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$ are $a'b', a'x_2'$.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$ are σ, $a'x_3'(=b'x_2')$ and $x_2'^2$.

The bordism invariants of $\Omega_5^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$ are $w_2w_3, a'b'x_2', a'x_2'^2, \mathfrak{P}_3(b')$.

Here $\mathfrak{P}_3$ is the Postnikov square.

i	$\mathrm{TP}_i(\mathrm{SO}\times\mathbb{Z}_3\times\mathrm{B}\mathbb{Z}_3)$
0	0
1	$\mathbb{Z}_3$
2	$\mathbb{Z}_3$
3	$\mathbb{Z}\times\mathbb{Z}_3^2$
4	$\mathbb{Z}_3^2$
5	$\mathbb{Z}_2\times\mathbb{Z}_3^2\times\mathbb{Z}_9$

Theorem 86.

The 2d topological term is x_2'.

The 3d topological terms are $\frac{1}{3}\mathrm{CS}_3^{(TM)}, a'b', a'x_2'$.

The 4d topological terms are $a'x_3'(=b'x_2')$ and $x_2'^2$.

The 5d topological terms are $w_2w_3, a'b'x_2', a'x_2'^2, \mathfrak{P}_3(b')$.

5.6.8 $\Omega_d^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}\wedge(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}. \tag{5.98}$$

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathbb{Z}_2$, we have $\Omega_d^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}=\Omega_d^{\mathrm{Spin}}$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}\wedge(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}. \tag{5.99}$$

Since

$$\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_3)=\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_3),$$

we have the following

i	$\Omega_i^{\text{Spin}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2\times\mathbb{Z}_3$
2	$\mathbb{Z}_2\times\mathbb{Z}_3$
3	$\mathbb{Z}_3^2$
4	$\mathbb{Z}\times\mathbb{Z}_3^2$
5	$\mathbb{Z}_3^2\times\mathbb{Z}_9$

Theorem 87.

The bordism invariants of $\Omega_2^{\text{Spin}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$ are Arf and x_2'.

The bordism invariants of $\Omega_3^{\text{Spin}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$ are $a'b', a'x_2'$.

The bordism invariants of $\Omega_4^{\text{Spin}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$ are $\frac{\sigma}{16}$, $a'x_3'(=b'x_2')$ and $x_2'^2$.

The bordism invariants of $\Omega_5^{\text{Spin}}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$ are $a'b'x_2', a'x_2'^2, \mathfrak{P}_3(b')$.

i	$\mathrm{TP}_i(\text{Spin}\times\mathbb{Z}_3\times\mathrm{B}\mathbb{Z}_3)$
0	0
1	$\mathbb{Z}_2\times\mathbb{Z}_3$
2	$\mathbb{Z}_2\times\mathbb{Z}_3$
3	$\mathbb{Z}\times\mathbb{Z}_3^2$
4	$\mathbb{Z}_3^2$
5	$\mathbb{Z}_3^2\times\mathbb{Z}_9$

Theorem 88.

The 2d topological terms are Arf and x_2'.

The 3d topological terms are $\frac{1}{48}\mathrm{CS}_3^{(TM)}, a'b', a'x_2'$.

The 4d topological terms are $a'x_3'(=b'x_2')$ and $x_2'^2$.

The 5d topological terms are $a'b'x_2', a'x_2'^2, \mathfrak{P}_3(b')$.

5.6.9 $\Omega_d^{\text{Pin}^+}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^-\wedge(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Pin}^+}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{5.100}$$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+\wedge(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\text{Pin}^-}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{5.101}$$

Since $\mathrm{H}^*(M\mathrm{Pin}^-,\mathbb{Z}_3)=0$, we have $\Omega_d^{\text{Pin}^+}(\mathrm{B}\mathbb{Z}_3\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge=0$.

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_2) = \mathbb{Z}_2$, we have $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge = \Omega_d^{\mathrm{Pin}^+}$.

Hence $\Omega_d^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3) = \Omega_d^{\mathrm{Pin}^+}$.

i	$\Omega_i^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_{16}$
5	0

Theorem 89.

The bordism invariant of $\Omega_2^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ is $w_1\tilde{\eta}$.

The bordism invariant of $\Omega_3^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ is $w_1\mathrm{Arf}$.

The bordism invariant of $\Omega_4^{\mathrm{Pin}^+}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$ is η.

i	$\mathrm{TP}_i(\mathrm{Pin}^+ \times \mathbb{Z}_3 \times \mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_{16}$
5	0

Theorem 90.

The 2d topological term is $w_1\tilde{\eta}$.

The 3d topological term is $w_1\mathrm{Arf}$.

The 4d topological term is η.

5.6.10 $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+ \wedge (\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{5.102}$$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Pin}^+ \wedge (\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{5.103}$$

Since $\mathrm{H}^*(M\mathrm{Pin}^+, \mathbb{Z}_3) = 0$, we have $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge = 0$.

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_2) = \mathbb{Z}_2$, we have $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge = \Omega_d^{\mathrm{Pin}^-}$.

Hence $\Omega_d^{\mathrm{Pin}^-}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3) = \Omega_d^{\mathrm{Pin}^-}$.

i	$\Omega_i^{\text{Pin}^-}(\text{B}\mathbb{Z}_3 \times \text{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	0
5	0

Theorem 91.

The bordism invariant of $\Omega_2^{\text{Pin}^-}(\text{B}\mathbb{Z}_3 \times \text{B}^2\mathbb{Z}_3)$ is ABK.

i	$\text{TP}_i(\text{Pin}^- \times \mathbb{Z}_3 \times \text{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	0
5	0

Theorem 92.

The 2d topological term is ABK.

5.7 $(\text{B}G_a, \text{B}^2G_b) : (\text{BPSU}(2), \text{B}^2\mathbb{Z}_2), (\text{BPSU}(3), \text{B}^2\mathbb{Z}_3)$

5.7.1 $\Omega_d^{\mathbf{O}}(\text{BPSU}(2) \times \text{B}^2\mathbb{Z}_2)$

$$\text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(M\text{O}, \mathbb{Z}_2) \otimes \text{H}^*(\text{BPSU}(2) \times \text{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{O}}(\text{BPSU}(2) \times \text{B}^2\mathbb{Z}_2). \tag{5.104}$$

$$\text{H}^*(\text{BPSU}(2), \mathbb{Z}_2) = \mathbb{Z}_2[w_2', w_3'], \tag{5.105}$$

$$\text{H}^*(\text{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots], \tag{5.106}$$

$$\text{H}^*(M\text{O}, \mathbb{Z}_2) = \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^*. \tag{5.107}$$

$$\begin{aligned} & \text{H}^*(M\text{O}, \mathbb{Z}_2) \otimes \text{H}^*(\text{BPSU}(2), \mathbb{Z}_2) \otimes \text{H}^*(\text{B}^2\mathbb{Z}_2, \mathbb{Z}_2) \\ = \; & \mathcal{A}_2 \oplus 3\Sigma^2\mathcal{A}_2 \oplus 2\Sigma^3\mathcal{A}_2 \oplus 7\Sigma^4\mathcal{A}_2 \oplus 8\Sigma^5\mathcal{A}_2 \oplus \cdots . \end{aligned} \tag{5.108}$$

Theorem 93.

i	$\Omega_i^{\mathrm{O}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_2^7$
5	$\mathbb{Z}_2^8$

The bordism invariants of $\Omega_2^{\mathrm{O}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$ are w_2', x_2, w_1^2.

The bordism invariants of $\Omega_3^{\mathrm{O}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$ are $x_3=w_1x_2, w_3'=w_1w_2'$.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$ are $w_1^4, w_2^2, x_2^2, w_2'^2, x_2w_1^2, w_2'w_1^2, w_2'x_2$.

The bordism invariants of $\Omega_5^{\mathrm{O}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$ are $w_2'w_3', x_2w_3', w_1^2w_3', w_2'x_3, x_2x_3, w_1^2x_3, x_5, w_2w_3$.

i	$\mathrm{TP}_i(\mathrm{O}\times \mathrm{PSU}(2)\times \mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_2^7$
5	$\mathbb{Z}_2^8$

Theorem 94.

The 2d topological terms are w_2', x_2, w_1^2.

The 3d topological terms are $x_3=w_1x_2, w_3'=w_1w_2'$.

The 4d topological terms are $w_1^4, w_2^2, x_2^2, w_2'^2, x_2w_1^2, w_2'w_1^2, w_2'x_2$.

The 5d topological terms are $w_2'w_3', x_2w_3', w_1^2w_3', w_2'x_3, x_2x_3, w_1^2x_3, x_5, w_2w_3$.

5.7.2 $\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2). \tag{5.109}$$

There is a differential d_2 corresponding to the Bockstein homomorphism $\beta_{(2,4)}:\mathrm{H}^*(-,\mathbb{Z}_4)\to \mathrm{H}^{*+1}(-,\mathbb{Z}_2)$ associated to $0\to\mathbb{Z}_2\to\mathbb{Z}_8\to\mathbb{Z}_4\to 0$ [57]. See 2.5 for the definition of Bockstein homomorphisms.

By (5.32), there is a differential such that $d_2(x_2x_3+x_5)=x_2^2h_0^2$.

The E_2 page is shown in Figure 59.

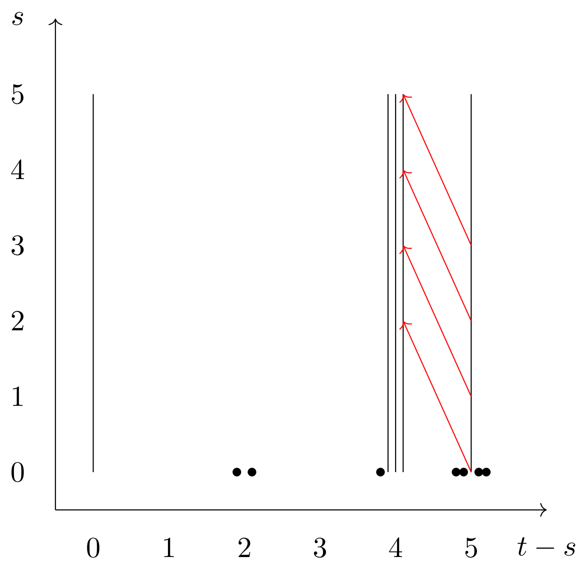


Figure 59: $\Omega_*^{\mathrm{SO}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_2^2$
3	0
4	$\mathbb{Z}^2\times\mathbb{Z}_2\times\mathbb{Z}_4$
5	$\mathbb{Z}_2^4$

Theorem 95.

The bordism invariants of $\Omega_2^{\mathrm{SO}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$ are w_2', x_2.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$ are σ, p_1', $w_2'x_2$ and $\mathcal{P}_2(x_2)$.

The bordism invariants of $\Omega_5^{\mathrm{SO}}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$ are $w_2'w_3', x_5, w_3'x_2(=w_2'x_3), w_2w_3$.

i	$\mathrm{TP}_i(\mathrm{SO}\times\mathrm{PSU}(2)\times \mathrm{B}\mathbb{Z}_2)$
0	0
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}^2$
4	$\mathbb{Z}_2\times\mathbb{Z}_4$
5	$\mathbb{Z}_2^4$

Theorem 96.

The 2d topological terms are w_2', x_2.

The 3d topological terms are $\frac{1}{3}\mathrm{CS}_3^{(TM)}$, $\mathrm{CS}_3^{(\mathrm{SO}(3))}$.

The 4d topological terms are $w_2'x_2$ and $\mathcal{P}_2(x_2)$.

The 5d topological terms are $w_2'w_3', x_5, w_3'x_2(=w_2'x_3), w_2w_3$.

5.7.3 $\Omega_d^{\text{Spin}}(\text{BPSU}(2)\times\text{B}^2\mathbb{Z}_2)$

For $t-s<8$,

$$\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(\text{H}^*(\text{BPSU}(2)\times\text{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Spin}}(\text{BPSU}(2)\times\text{B}^2\mathbb{Z}_2). \quad (5.110)$$

There is a differential d_2 corresponding to the Bockstein homomorphism $\beta_{(2,4)}:\text{H}^*(-,\mathbb{Z}_4)\to\text{H}^{*+1}(-,\mathbb{Z}_2)$ associated to $0\to\mathbb{Z}_2\to\mathbb{Z}_8\to\mathbb{Z}_4\to 0$ [57]. See 2.5 for the definition of Bockstein homomorphisms.

By (5.32), there is a differential such that $d_2(x_2x_3+x_5)=x_2^2h_0^2$.

The $\mathcal{A}_2(1)$-module structure of $\text{H}^*(\text{BPSU}(2)\times\text{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ and the E_2 page is shown in Figure 60, 61.

i	$\Omega_i^{\text{Spin}}(\text{BPSU}(2)\times\text{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	0
4	$\mathbb{Z}^2\times\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 97.

The bordism invariants of $\Omega_2^{\text{Spin}}(\text{BPSU}(2)\times\text{B}^2\mathbb{Z}_2)$ are w_2', x_2, Arf.

The bordism invariants of $\Omega_4^{\text{Spin}}(\text{BPSU}(2)\times\text{B}^2\mathbb{Z}_2)$ are $\frac{\sigma}{16}$, $\frac{p_1'}{2}$, $w_2'x_2$ and $\frac{\mathcal{P}_2(x_2)}{2}$.

The bordism invariant of $\Omega_5^{\text{Spin}}(\text{BPSU}(2)\times\text{B}^2\mathbb{Z}_2)$ is $w_3'x_2(=w_2'x_3)$.

i	$\text{TP}_i(\text{Spin}\times\text{PSU}(2)\times\text{B}\mathbb{Z}_2)$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}^2$
4	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_2$

Theorem 98.

The 2d topological terms are w_2', x_2, Arf.

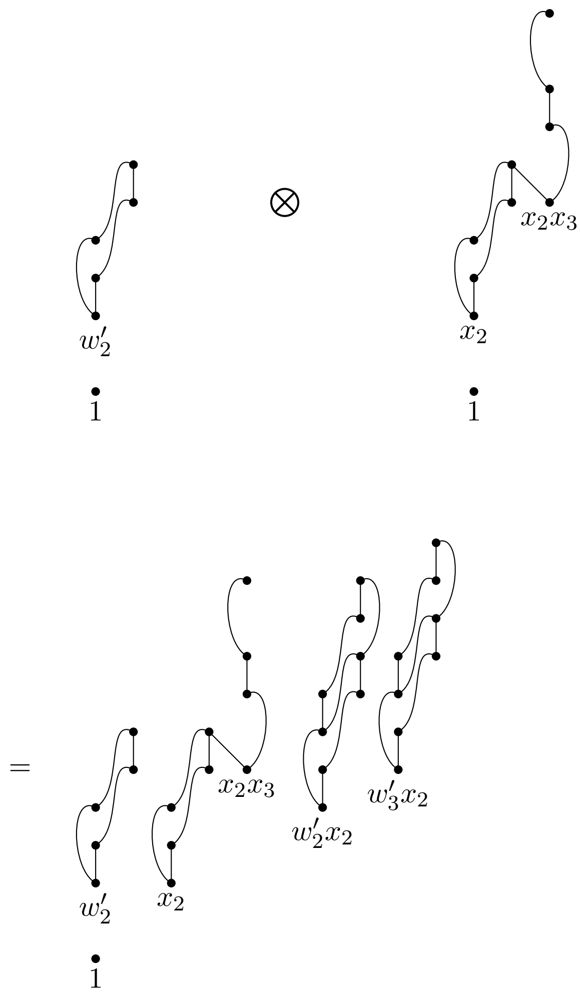


Figure 60: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^*(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$

The 3d topological terms are $\frac{1}{48}\mathrm{CS}_3^{(TM)}$, $\frac{1}{2}\mathrm{CS}_3^{(\mathrm{SO}(3))}$.

The 4d topological terms are $w_2'x_2$ and $\frac{\mathcal{P}_2(x_2)}{2}$.

The 5d topological term is $w_3'x_2 (= w_2'x_3)$.

5.7.4 $\Omega_d^{\mathrm{Pin}^+}(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2)$

For $t - s < 8$,

$$\begin{aligned} &\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*-1}(MT\mathrm{O}(1), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \\ \Rightarrow\ &\Omega_{t-s}^{\mathrm{Pin}^+}(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2). \end{aligned} \tag{5.111}$$

The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1), \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{BPSU}(2) \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$ and the E_2 page are shown in Figure 62, 63.

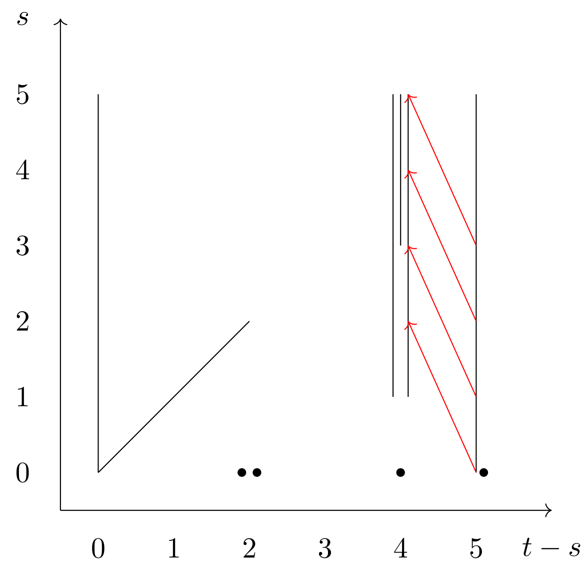


Figure 61: $\Omega_*^{\mathrm{Spin}}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{Pin}^+}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^3$
4	$\mathbb{Z}_4^2\times\mathbb{Z}_{16}\times\mathbb{Z}_2$
5	$\mathbb{Z}_2^5$

Theorem 99.

The bodism invariants of $\Omega_2^{\mathrm{Pin}^+}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$ are $w_2', x_2, w_1\tilde{\eta}$.

The bodism invariants of $\Omega_3^{\mathrm{Pin}^+}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$ are $w_1w_2'=w_3', w_1x_2=x_3, w_1\mathrm{Arf}$.

The bodism invariants of $\Omega_4^{\mathrm{Pin}^+}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$ are $q_s(w_2'), q_s(x_2), \eta, w_2'x_2$.

The bodism invariants of $\Omega_5^{\mathrm{Pin}^+}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$ are

$$w_1^2w_3'(=w_2'w_3'), w_1^2x_3(=x_5), x_2x_3, w_3'x_2, w_1w_2'x_2(=w_2'x_3+w_3'x_2).$$

i	$\mathrm{TP}_i(\mathrm{Pin}^+\times\mathrm{PSU}(2)\times\mathrm{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^3$
4	$\mathbb{Z}_4^2\times\mathbb{Z}_{16}\times\mathbb{Z}_2$
5	$\mathbb{Z}_2^5$

Theorem 100.

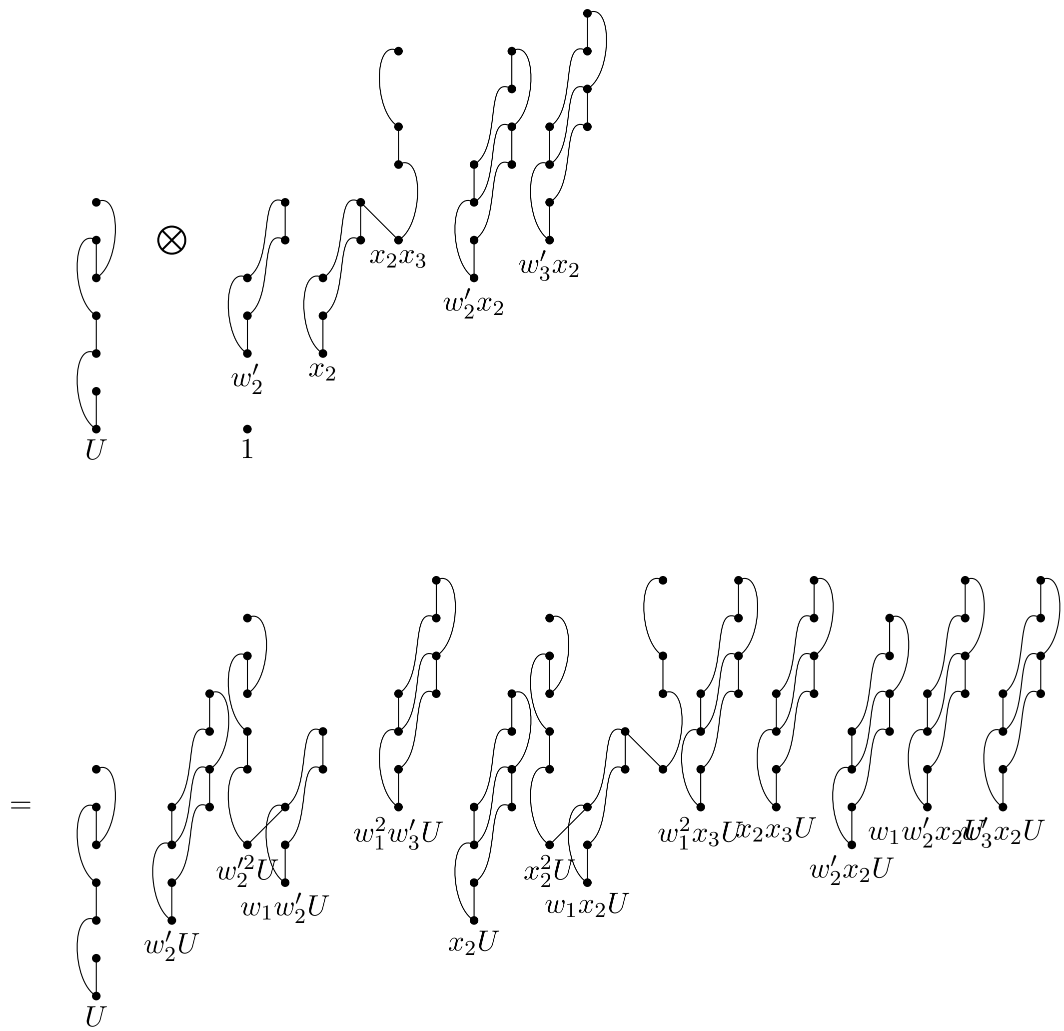


Figure 62: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*-1}(MT\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$

The 2d topological terms are $w_2', x_2, w_1\tilde{\eta}$.

The 3d topological terms are $w_1w_2' = w_3', w_1x_2 = x_3, w_1\mathrm{Arf}$.

The 4d topological terms are $q_s(w_2'), q_s(x_2), \eta, w_2'x_2$.

The 5d topological terms are

$$w_1^2w_3'(=w_2'w_3'), w_1^2x_3(=x_5), x_2x_3, w_3'x_2, w_1w_2'x_2(=w_2'x_3+w_3'x_2).$$

5.7.5 $\Omega_d^{\mathrm{Pin}^-}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2)$

For $t-s<8$,

$$\mathrm{Ext}_{\mathcal{A}_2(1)}^{s,t}(\mathrm{H}^{*+1}(M\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Pin}^-}(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2).\ (5.112)$$

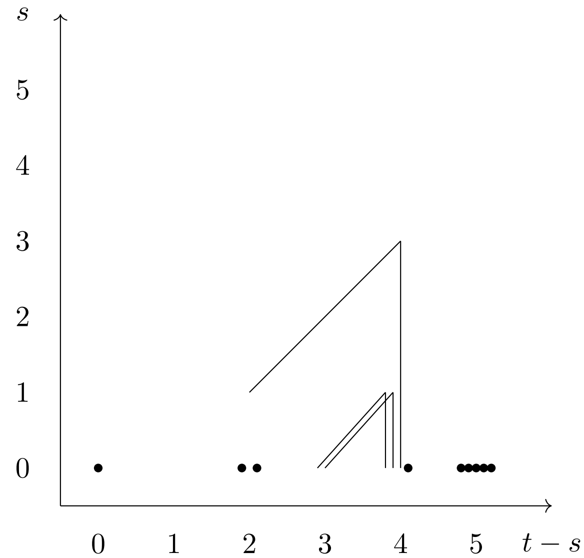


Figure 63: $\Omega_*^{\text{Pin}^+}(\text{BPSU}(2)\times \text{B}^2\mathbb{Z}_2)$

The $\mathcal{A}_2(1)$-module structure of $\text{H}^{*+1}(M\text{O}(1).\mathbb{Z}_2)\otimes \text{H}^*(\text{BPSU}(2)\times \text{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$ and the E_2 page are shown in Figure 64, 65.

i	$\Omega_i^{\text{Pin}^-}(\text{BPSU}(2)\times \text{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2\times\mathbb{Z}_8$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2^3$

Theorem 101.

The bordism invariants of $\Omega_2^{\text{Pin}^-}(\text{BPSU}(2)\times \text{B}^2\mathbb{Z}_2)$ are w_2', x_2, ABK.

The bordism invariants of $\Omega_3^{\text{Pin}^-}(\text{BPSU}(2)\times \text{B}^2\mathbb{Z}_2)$ are $w_1w_2'=w_3', w_1x_2=x_3$.

The bordism invariants of $\Omega_4^{\text{Pin}^-}(\text{BPSU}(2)\times \text{B}^2\mathbb{Z}_2)$ are $w_1^2w_2', w_1^2x_2, w_2'x_2$.

The bordism invariants of $\Omega_5^{\text{Pin}^-}(\text{BPSU}(2)\times \text{B}^2\mathbb{Z}_2)$ are $x_2x_3, w_3'x_2, w_1w_2'x_2(=w_2'x_3+w_3'x_2)$.

i	$\text{TP}_i(\text{Pin}^-\times\text{PSU}(2)\times \text{B}\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2\times\mathbb{Z}_8$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2^3$

Theorem 102.

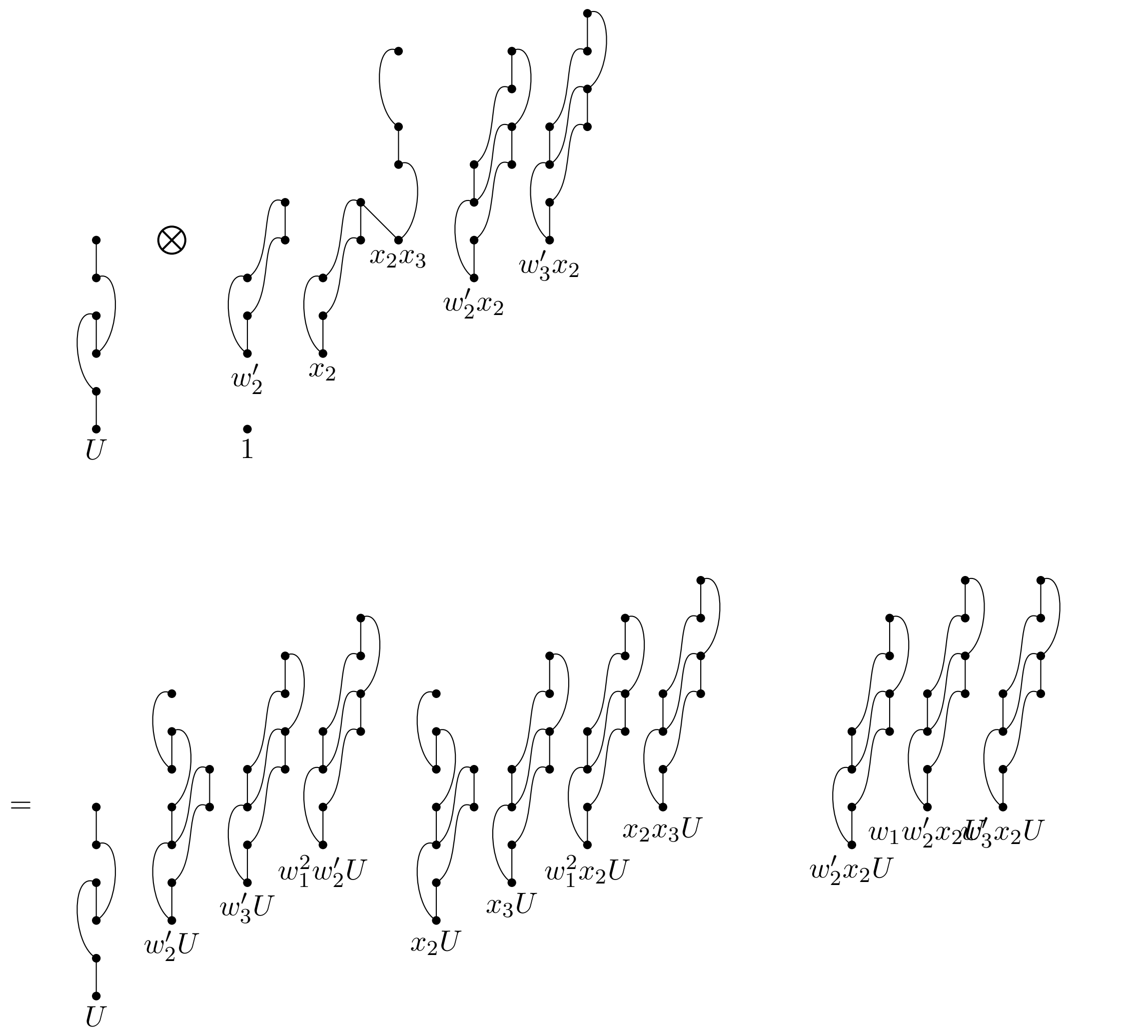


Figure 64: The $\mathcal{A}_2(1)$-module structure of $\mathrm{H}^{*+1}(M\mathrm{O}(1),\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BPSU}(2)\times\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$

The 2d topological terms are w_2', x_2, ABK.

The 3d topological terms are $w_1w_2' = w_3', w_1x_2 = x_3$.

The 4d topological terms are $w_1^2w_2', w_1^2x_2, w_2'x_2$.

The 5d topological terms are $x_2x_3, w_3'x_2, w_1w_2'x_2 (= w_2'x_3 + w_3'x_2)$.

5.7.6 $\Omega_d^{\mathrm{O}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{O}\wedge(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}. \tag{5.113}$$

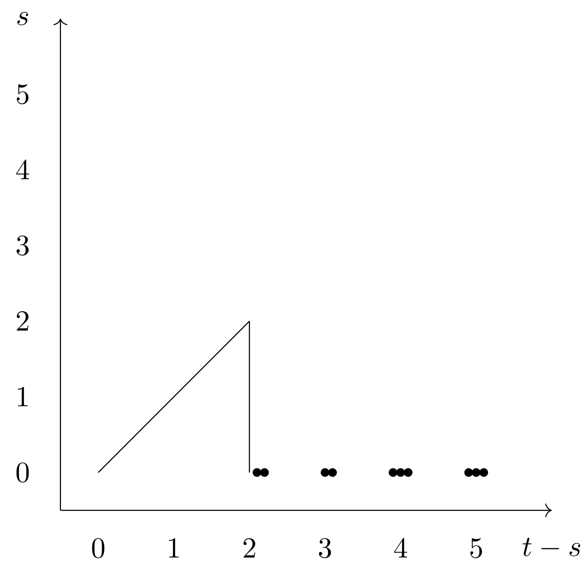


Figure 65: $\Omega_*^{\mathrm{Pin}^-}(\mathrm{BPSU}(2)\times \mathrm{B}^2\mathbb{Z}_2)$

Since $\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_3)=0$, $\Omega_d^{\mathrm{O}}(\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge=0$.

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O}\wedge(\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{5.114}$$

Since $\mathrm{H}^*(\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2)$, $\Omega_d^{\mathrm{O}}(\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\mathrm{O}}(\mathrm{BPSU}(3))_2^\wedge$.

i	$\Omega_i^{\mathrm{O}}(\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2$

Theorem 103.

The bordism invariant of $\Omega_2^{\mathrm{O}}(\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3)$ is w_1^2.

The bordism invariants of $\Omega_4^{\mathrm{O}}(\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3)$ are $w_1^4, w_2^2, c_2(\mod 2)$.

The bordism invariant of $\Omega_5^{\mathrm{O}}(\mathrm{BPSU}(3)\times \mathrm{B}^2\mathbb{Z}_3)$ is w_2w_3.

Theorem 104.

The 2d topological term is w_1^2.

The 4d topological terms are $w_1^4, w_2^2, c_2(\mod 2)$.

The 5d topological term is w_2w_3.

i	$\mathrm{TP}_i(\mathrm{O}\times\mathrm{PSU}(3)\times\mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}_2^3$
5	$\mathbb{Z}_2$

5.7.7 $\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{5.115}$$

Since $\mathrm{H}^*(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2)$, $\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(3))_2^\wedge$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{5.116}$$

$\beta_{(3,3)}x_2'=x_3'$, $\beta_{(3,3)}z_2=z_3$, $\beta_{(3,3)}x_2'^2=2x_2'x_3'$, $\beta_{(3,3)}(x_2'z_2)=x_2'z_3+x_3'z_2$, $\beta_{(3,3)}(x_2'z_3)=x_3'z_3=-\beta_{(3,3)}(x_3'z_2)$.

The E_2 page is shown in Figure 66.

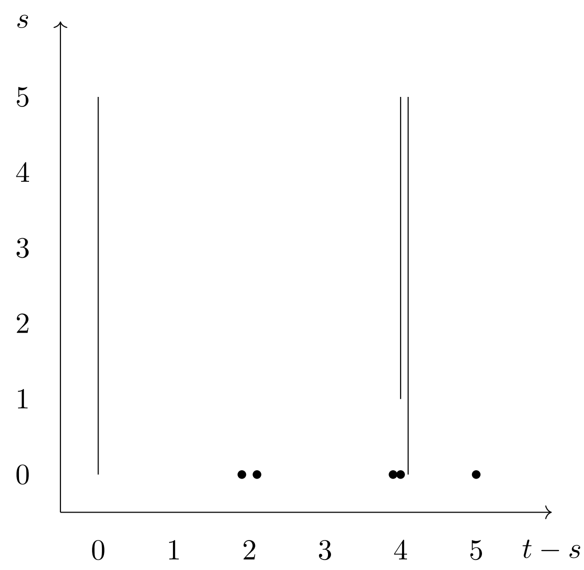


Figure 66: $\Omega_*^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge$

Theorem 105.

The bordism invariants of $\Omega_2^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ are x_2',z_2.

The bordism invariants of $\Omega_4^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ are σ, c_2, $x_2'^2$ and $x_2'z_2$.

The bordism invariants of $\Omega_5^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ are $w_2w_3, z_2x_3'(=-z_3x_2')$.

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_3^2$
3	0
4	$\mathbb{Z}^2\times\mathbb{Z}_3^2$
5	$\mathbb{Z}_2\times\mathbb{Z}_3$

i	$\mathrm{TP}_i(\mathrm{SO}\times\mathrm{PSU}(3)\times\mathrm{B}\mathbb{Z}_3)$
0	0
1	0
2	$\mathbb{Z}_3^2$
3	$\mathbb{Z}^2$
4	$\mathbb{Z}_3^2$
5	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_3$

Theorem 106.

The 2d topological terms are x_2', z_2.

The 3d topological terms are $\frac{1}{3}\mathrm{CS}_3^{(TM)}$, $\mathrm{CS}_3^{(\mathrm{PSU}(3))}$.

The 4d topological terms are $x_2'^2$ and $x_2'z_2$.

The 5d topological terms are $\mathrm{CS}_5^{(\mathrm{PSU}(3))}, w_2w_3, z_2x_3'(=-z_3x_2')$.

5.7.8 $\Omega_d^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}\wedge(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}. \quad (5.117)$$

Since $\mathrm{H}^*(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2)$, $\Omega_d^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_2^{\wedge}=\Omega_d^{\mathrm{Spin}}(\mathrm{BPSU}(3))_2^{\wedge}$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{Spin}\wedge(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}. \quad (5.118)$$

Since $\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_3)=\mathrm{H}^*(M\mathrm{Spin},\mathbb{Z}_3)$, $\Omega_d^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}=\Omega_d^{\mathrm{SO}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)_3^{\wedge}$.

Theorem 107.

The bordism invariants of $\Omega_2^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ are Arf, x_2', z_2.

The bordism invariants of $\Omega_4^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ are $\frac{\sigma}{16}$, c_2, $x_2'^2$ and $x_2'z_2$.

The bordism invariant of $\Omega_5^{\mathrm{Spin}}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ is $z_2x_3'(=-z_3x_2')$.

i	$\Omega_i^{\text{Spin}}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2\times\mathbb{Z}_3^2$
3	0
4	$\mathbb{Z}^2\times\mathbb{Z}_3^2$
5	$\mathbb{Z}_3$

i	$\text{TP}_i(\text{Spin}\times\text{PSU}(3)\times\text{B}\mathbb{Z}_3)$
0	0
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2\times\mathbb{Z}_3^2$
3	$\mathbb{Z}^2$
4	$\mathbb{Z}_3^2$
5	$\mathbb{Z}\times\mathbb{Z}_3$

Theorem 108.

The 2d topological terms are Arf, x_2', z_2.

The 3d topological terms are $\frac{1}{48}\text{CS}_3^{(TM)}$, $\text{CS}_3^{(\text{PSU}(3))}$.

The 4d topological terms are $x_2'^2$ and $x_2'z_2$.

The 5d topological terms are $\frac{1}{2}\text{CS}_5^{(\text{PSU}(3))}, z_2x_3'(=-z_3x_2')$.

5.7.9 $\Omega_d^{\text{Pin}^+}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)$

$$\text{Ext}_{\mathcal{A}_3}^{s,t}(\text{H}^*(M\text{Pin}^-\wedge(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\text{Pin}^+}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)_3^\wedge. \tag{5.119}$$

Since $\text{H}^*(M\text{Pin}^-,\mathbb{Z}_3)=0$, $\Omega_d^{\text{Pin}^+}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)_3^\wedge=0$.

$$\text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(M\text{Pin}^-\wedge(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Pin}^+}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)_2^\wedge. \tag{5.120}$$

Since $\text{H}^*(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2)$, $\Omega_d^{\text{Pin}^+}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\text{Pin}^+}(\text{BPSU}(3))_2^\wedge$.

Theorem 109.

The bordism invariant of $\Omega_2^{\text{Pin}^+}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)$ is $w_1\tilde\eta$.

The bordism invariant of $\Omega_3^{\text{Pin}^+}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)$ is $w_1\text{Arf}$.

The bordism invariants of $\Omega_4^{\text{Pin}^+}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)$ are $c_2(\mod 2)$ and η.

i	$\Omega_i^{\text{Pin}^+}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2\times\mathbb{Z}_{16}$
5	0

i	$\text{TP}_i(\text{Pin}^+\times\text{PSU}(3)\times\text{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2\times\mathbb{Z}_{16}$
5	0

Theorem 110.

The 2d topological term is $w_1\tilde{\eta}$.

The 3d topological term is $w_1\text{Arf}$.

The 4d topological terms are $c_2(\mod 2)$ and η.

5.7.10 $\Omega_d^{\text{Pin}^-}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)$

$$\text{Ext}_{\mathcal{A}_3}^{s,t}(\text{H}^*(M\text{Pin}^+\wedge(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\text{Pin}^-}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)_3^\wedge. \tag{5.121}$$

Since $\text{H}^*(M\text{Pin}^+,\mathbb{Z}_3)=0$, $\Omega_d^{\text{Pin}^-}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)_3^\wedge=0$.

$$\text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(M\text{Pin}^+\wedge(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{Pin}^-}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)_2^\wedge. \tag{5.122}$$

Since $\text{H}^*(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\text{H}^*(\text{BPSU}(3),\mathbb{Z}_2)$, $\Omega_d^{\text{Pin}^-}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\text{Pin}^-}(\text{BPSU}(3))_2^\wedge$.

i	$\Omega_i^{\text{Pin}^-}(\text{BPSU}(3)\times\text{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	$\mathbb{Z}_2$
5	0

Theorem 111.

The bordism invariant of $\Omega_2^{\mathrm{Pin}^-}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ is ABK.

The bordism invariant of $\Omega_4^{\mathrm{Pin}^-}(\mathrm{BPSU}(3)\times\mathrm{B}^2\mathbb{Z}_3)$ is $c_2(\mod 2)$.

i	$\mathrm{TP}_i(\mathrm{Pin}^-\times\mathrm{PSU}(3)\times\mathrm{B}\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_8$
3	0
4	$\mathbb{Z}_2$
5	0

Theorem 112.

The 2d topological term is ABK.

The 4d topological term is $c_2(\mod 2)$.

6 More computation of O/SO bordism groups

6.1 Summary

Below we use the following notations, all cohomology class are pulled back to the d-manifold M along the maps given in the definition of cobordism groups:
• w_i is the Stiefel-Whitney class of the tangent bundle of M,
• a is the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)$,
• a' is the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3)$, $b'=\beta_{(3,3)}a'$,
• x_2 is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)$, $x_3=\mathrm{Sq}^1x_2$, $x_5=\mathrm{Sq}^2x_3$,
• x_2' is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_3)$, $x_3'=\beta_{(3,3)}x_2'$,
• x_2'' is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_4,\mathbb{Z}_4)$, $x_3''=\beta_{(2,4)}x_2''$, $x_5''=\mathrm{Sq}^2x_3''$,
• $w_i'=w_i(\mathrm{O}(n))\in\mathrm{H}^i(\mathrm{BO}(n),\mathbb{Z}_2)$ is the Stiefel-Whitney class of the principal $\mathrm{O}(n)$ bundle,
• $p_1'=p_1(\mathrm{O}(n))\in\mathrm{H}^4(\mathrm{BO}(n),\mathbb{Z}_2)$ is the first Pontryagin class of the principal $\mathrm{O}(n)$ bundle,
• $z_2=w_2(\mathrm{PSU}(3))\in\mathrm{H}^2(\mathrm{BPSU}(3),\mathbb{Z}_3)$ is the generalized Stiefel-Whitney class of the principal $\mathrm{PSU}(3)$ bundle, $z_3=\beta_{(3,3)}z_2$.
• $z_2'=w_2(\mathrm{PSU}(4))\in\mathrm{H}^2(\mathrm{BPSU}(4),\mathbb{Z}_4)$ is the generalized Stiefel-Whitney class of the principal $\mathrm{PSU}(4)$ bundle, $z_3'=\beta_{(2,4)}z_2'$.
• For $n>1$, we also use the notation a for the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_{2^n},\mathbb{Z}_{2^n})$, $\tilde{a}=a\mod 2$, b is the generator of $\mathrm{H}^2(\mathrm{B}\mathbb{Z}_{2^n},\mathbb{Z}_{2^n})$, $\tilde{b}=b\mod 2$ and $\tilde{b}=\beta_{(2,2^n)}a$.
• For $n>1$, we also use the notation a' for the generator of $\mathrm{H}^1(\mathrm{B}\mathbb{Z}_{3^n},\mathbb{Z}_{3^n})$, $\tilde{a}'=a'\mod 3$, b' is the generator of $\mathrm{H}^2(\mathrm{B}\mathbb{Z}_{3^n},\mathbb{Z}_{3^n})$, $\tilde{b}'=b'\mod 3$ and $\tilde{b}'=\beta_{(3,3^n)}a'$.
• $\mathcal{P}_2$ is the Pontryagin square (see 1.5).
• $\mathfrak{P}_3$ is the Postnikov square (see 1.5).
Convention: All product between cohomology classes are cup product.

$\Omega_d^H(-)$	BO(3)	BO(4)	BO(5)	$\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3))$	$\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4))$
2 SO	$\mathbb{Z}_2$: w_2'	$\mathbb{Z}_2$: w_2'	$\mathbb{Z}_2$: w_2'	$\mathbb{Z}_3$: z_2	$\mathbb{Z}_4$: z_2'
2 O	$\mathbb{Z}_2^3$: w_1^2, $w_1'^2, w_2'$	$\mathbb{Z}_2^3$: w_1^2, $w_1'^2, w_2'$	$\mathbb{Z}_2^3$: w_1^2, $w_1'^2, w_2'$	$\mathbb{Z}_2^2$: w_1^2, a^2	$\mathbb{Z}_2^2$: w_1^2, a^2, $\tilde{z}_2'$

Table 13: $2d$ bordism groups-1.

$\Omega_d^H(-)$	$\mathrm{B}^2\mathbb{Z}_4$	$\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3$	$\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3$
2 SO	$\mathbb{Z}_4$: x_2''	$\mathbb{Z}_2$: x_2	$\mathbb{Z}_3$: x_2'	$\mathbb{Z}_2$: x_2	$\mathbb{Z}_3$: x_2'
2 O	$\mathbb{Z}_2^2$: $w_1^2, \tilde{x}_2''$	$\mathbb{Z}_2^3$: $w_1^2, \tilde{b}$, x_2	$\mathbb{Z}_2^2$: w_1^2, a^2	$\mathbb{Z}_2^3$: $w_1^2, \tilde{b}$, x_2	$\mathbb{Z}_2^2$: w_1^2, a^2

Table 14: $2d$ bordism groups-2.

$\Omega_d^H(-)$	BO(3)	BO(4)	BO(5)	$\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3))$	$\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4))$
3 SO	$\mathbb{Z}_2^2$: $w_1'^3$, $w_1'w_2' = w_3'$	$\mathbb{Z}_2^2$: $w_1'^3$, $w_1'w_2' = w_3'$	$\mathbb{Z}_2^2$: $w_1'^3$, $w_1'w_2' = w_3'$	$\mathbb{Z}_2$: a^3	$\mathbb{Z}_2^2$: $a^3, a\tilde{z}_2'$
3 O	$\mathbb{Z}_2^4$: $w_1'w_1^2, w_1'^3$, $w_1'w_2', w_3'$	$\mathbb{Z}_2^4$: $w_1'w_1^2, w_1'^3$, $w_1'w_2', w_3'$	$\mathbb{Z}_2^4$: $w_1'w_1^2, w_1'^3$, $w_1'w_2', w_3'$	$\mathbb{Z}_2^2$: a^3, aw_1^2	$\mathbb{Z}_2^4$: a^3, aw_1^2, $z_3', a\tilde{z}_2'$

Table 15: $3d$ bordism groups-1.

$\Omega_d^H(-)$	$\mathrm{B}^2\mathbb{Z}_4$	$\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3$	$\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3$
3 SO	0	$\mathbb{Z}_4\times\mathbb{Z}_2$: $ab, \tilde{a}x_2$	$\mathbb{Z}_3^2\times\mathbb{Z}_2$: $a'b', a'x_2'$, a^3	$\mathbb{Z}_8\times\mathbb{Z}_2$: $ab, \tilde{a}x_2$	$\mathbb{Z}_9 \times \mathbb{Z}_3\times\mathbb{Z}_2$: $a'b', \tilde{a}'x_2'$, a^3
3 O	$\mathbb{Z}_2$: x_3''	$\mathbb{Z}_2^4$: $\tilde{a}\tilde{b}, x_3$, $\tilde{a}x_2, \tilde{a}w_1^2$	$\mathbb{Z}_2^2$: a^3, aw_1^2	$\mathbb{Z}_2^4$: $\tilde{a}\tilde{b}, x_3$, $\tilde{a}x_2, \tilde{a}w_1^2$	$\mathbb{Z}_2^2$: a^3, aw_1^2

Table 16: $3d$ bordism groups-2.

6.2 $\mathrm{B}^2\mathbb{Z}_4$

6.2.1 $\Omega_d^{\mathbf{O}}(\mathrm{B}^2\mathbb{Z}_4)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_4,\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_4) \tag{6.1}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2) = \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \tag{6.2}$$

where $y_2^* = w_1^2$, $(y_2^2)^* = w_2^2$, $y_4^* = w_1^4$, $y_5^* = w_2w_3$, etc.

$\Omega_d^H(-)$	BO(3)	BO(4)	BO(5)
4 SO	$\mathbb{Z}^2\times\mathbb{Z}_2$: σ, p'_1, $w'^2_1 w'_2$	$\mathbb{Z}^2\times\mathbb{Z}_2^2$: σ, p'_1, $w'^2_1 w'_2, w'_4$	$\mathbb{Z}^2\times\mathbb{Z}_2^2$: σ, p'_1, $w'^2_1 w'_2, w'_4$
4 O	$\mathbb{Z}_2^8$: w_1^4, w_2^2, $w_1^2 w'^2_1, w_1^2 w'_2$, $w'_1 w'_3, w'^2_1 w'_2$, w'^4_1, w'^2_2	$\mathbb{Z}_2^8$: w_1^4, w_2^2, $w_1^2 w'^2_1, w_1^2 w'_2$, $w'_1 w'_3, w'^2_1 w'_2$, w'^4_1, w'^2_2, w'_4	$\mathbb{Z}_2^8$: w_1^4, w_2^2, $w_1^2 w'^2_1, w_1^2 w'_2$, $w'_1 w'_3, w'^2_1 w'_2$, w'^4_1, w'^2_2, w'_4

Table 17: $4d$ bordism groups-1.

$\Omega_d^H(-)$	$\mathrm{B}^2\mathbb{Z}_4$	$\mathrm{B}\mathbb{Z}_4\times\mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3$	$\mathrm{B}\mathbb{Z}_8\times\mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3$
4 SO	$\mathbb{Z}\times\mathbb{Z}_8$: $\sigma, \mathcal{P}_2(x''_2)$	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$: $\sigma, \mathcal{P}_2(x_2)$, $\tilde{b}x_2$	$\mathbb{Z}\times\mathbb{Z}_3^2$: $\sigma, a'x'_3 = b'x'_2$, x'^2_2	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$: $\sigma, \mathcal{P}_2(x_2)$, $\tilde{b}x_2$	$\mathbb{Z}\times\mathbb{Z}_3^2$: $\sigma, \tilde{b}'x'_2$, x'^2_2
4 O	$\mathbb{Z}_2^4$: w_1^4, w_2^2, $w_1^2\tilde{x}''_2, \tilde{x}''^2_2$	$\mathbb{Z}_2^8$: $\tilde{a}x_3, \tilde{b}x_2$, $\tilde{b}^2, x_2^2$, w_1^4, w_2^2, $\tilde{b}w_1^2, x_2w_1^2$	$\mathbb{Z}_2^4$: w_1^4, w_2^2, $a^4, a^2w_1^2$	$\mathbb{Z}_2^8$: $\tilde{a}x_3, \tilde{b}x_2$, $\tilde{b}^2, x_2^2$, w_1^4, w_2^2, $\tilde{b}w_1^2, x_2w_1^2$	$\mathbb{Z}_2^4$: w_1^4, w_2^2, $a^4, a^2w_1^2$

Table 18: $4d$ bordism groups-2.

$\Omega_d^H(-)$	BO(3)	BO(4)	BO(5)
5 SO	$\mathbb{Z}_2^6$: $w_2w_3, w_2^2w'_1$, $w'_2w'_3, w'_1w'^2_2$, $w'^2_1w'_3 =$ $w'^3_1w'_2, w'^5_1$	$\mathbb{Z}_2^6$: $w_2w_3, w_2^2w'_1$, $w'_2w'_3, w'_1w'^2_2$, $w'^2_1w'_3 =$ $w'^3_1w'_2, w'^5_1$	$\mathbb{Z}_2^7$: $w_2w_3, w_2^2w'_1$, $w'_2w'_3, w'_1w'^2_2$, $w'^2_1w'_3 =$ $w'^3_1w'_2, w'^5_1$, $w'_1w'_4 = w'_5$
5 O	$\mathbb{Z}_2^{11}$: $w_2w_3, w_2^2w'_1$, $w_1^4w'_1, w_1^2w'^3_1$, $w_1^2w'_1w'_2, w_1^2w'_3$, $w'_2w'_3, w'_1w'^2_2$, $w'^2_1w'_3, w'^3_1w'_2$, w'^5_1	$\mathbb{Z}_2^{12}$: $w_2w_3, w_2^2w'_1$, $w_1^4w'_1, w_1^2w'^3_1$, $w_1^2w'_1w'_2, w_1^2w'_3$, $w'_2w'_3, w'_1w'^2_2$, $w'^2_1w'_3, w'^3_1w'_2$, $w'^5_1, w'_1w'_4$	$\mathbb{Z}_2^{13}$: $w_2w_3, w_2^2w'_1$, $w_1^4w'_1, w_1^2w'^3_1$, $w_1^2w'_1w'_2, w_1^2w'_3$, $w'_2w'_3, w'_1w'^2_2$, $w'^2_1w'_3, w'^3_1w'_2$, $w'^5_1, w'_1w'_4$, w'_5

Table 19: $5d$ bordism groups-1.

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_2) = \mathbb{Z}_2[\tilde{x}''_2, x''_3, x''_5, x''_9, \dots] \tag{6.3}$$

where $\tilde{x}''_2 = x''_2 \mod 2$, $x''_2 \in \mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_4)$, $x''_3 = \beta_{(2,4)}x''_2$, $x''_5 = \mathrm{Sq}^2 x''_3$, $x''_9 = \mathrm{Sq}^4 x''_5$, etc.

$\Omega_d^H(-)$	$\mathrm{B}^2\mathbb{Z}_4$	$\mathrm{B}\mathbb{Z}_4\times\mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3$	$\mathrm{B}\mathbb{Z}_8\times\mathrm{B}^2\mathbb{Z}_2$	$\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3$
5 SO	$\mathbb{Z}_2^2$: w_2w_3, x_5''	$\mathbb{Z}_4^3\times\mathbb{Z}_2^3$: $a\mathcal{P}_2(x_2), ab^2$, $a(\sigma \bmod 4), x_5 = x_2x_3$, $\tilde{a}\tilde{b}x_2, w_2w_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3^2\times\mathbb{Z}_9$: a^5, aw_2^2, $w_2w_3, a'b'x_2'$ $a'x_2'^2, \mathfrak{P}_3(b')$	$\mathbb{Z}_4\times\mathbb{Z}_8^2\times\mathbb{Z}_2^3$: $(a \bmod 4)\mathcal{P}_2(x_2), ab^2$, $a(\sigma \bmod 8), x_5 = x_2x_3$, $\tilde{a}\tilde{b}x_2, w_2w_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3^3\times\mathbb{Z}_{27}$: a^5, aw_2^2, $w_2w_3, \tilde{a}'(\sigma \bmod 3)$, $\tilde{a}'\tilde{b}'x_2', \tilde{a}'x_2'^2$, $\mathfrak{P}_3(b')$
5 O	$\mathbb{Z}_2^4$: $w_2w_3, w_1^2x_3''$, $\tilde{x}_2''x_3'', x_5''$	$\mathbb{Z}_2^{12}$: $\tilde{a}x_2^2, \tilde{b}x_3$, $x_2x_3, \tilde{a}\tilde{b}^2$, $x_5, \tilde{a}\tilde{b}x_2$, $w_2w_3, \tilde{a}w_2^2$, $\tilde{a}w_1^4, \tilde{a}\tilde{b}w_1^2$, $x_3w_1^2 = w_1^3x_2, \tilde{a}x_2w_1^2$	$\mathbb{Z}_2^5$: $a^5, a^3w_1^2$, aw_1^4, aw_2^2, w_2w_3	$\mathbb{Z}_2^{12}$: $\tilde{a}x_2^2, \tilde{b}x_3$, $x_2x_3, \tilde{a}\tilde{b}^2$, $x_5, \tilde{a}\tilde{b}x_2$, $w_2w_3, \tilde{a}w_2^2$, $\tilde{a}w_1^4, \tilde{a}\tilde{b}w_1^2$, $x_3w_1^2 = w_1^3x_2, \tilde{a}x_2w_1^2$	$\mathbb{Z}_2^5$: $a^5, a^3w_1^2$, aw_1^4, aw_2^2, w_2w_3

Table 20: $5d$ bordism groups-2.

$$\begin{aligned}\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_4,\mathbb{Z}_2) &= \mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^*\otimes\mathbb{Z}_2[\tilde{x}_2'',x_3'',x_5'',x_9'',\dots] \\ &= \mathcal{A}_2\oplus 2\Sigma^2\mathcal{A}_2\oplus\Sigma^3\mathcal{A}_2\oplus 4\Sigma^4\mathcal{A}_2\oplus 4\Sigma^5\mathcal{A}_2\oplus\cdots\end{aligned} \tag{6.4}$$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_4)$
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^4$

Theorem 113.

The 2d bordism invariants are $w_1^2, \tilde{x}_2''$.

The 3d bordism invariant is x_3''.

The 4d bordism invariants are $w_1^4, w_2^2, w_1^2\tilde{x}_2'', \tilde{x}_2''^2$.

The 5d bordism invariants are $w_2w_3, w_1^2x_3'', \tilde{x}_2''x_3'', x_5''$.

6.2.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_4)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_4,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_4) \tag{6.5}$$

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)=\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^5\mathcal{A}_2\oplus\cdots \tag{6.6}$$

Note that $\mathrm{Sq}^1\tilde{x}''_2=2\beta_{(2,4)}x''_2=0$, $\beta_{(2,4)}(x''_2)=\frac{1}{4}\delta x''_2=x''_3$, $\beta_{(2,4)}(x''^2_2)=2x''_2x''_3=2\tilde{x}''_2x''_3=0$, $\mathrm{Sq}^1(\tilde{x}''^2_2)=2\beta_{(2,4)}(x''^2_2)=0$, $\mathrm{Sq}^1x''_3=0$, $\mathrm{Sq}^1(\tilde{x}''_2x''_3)=0$, $\mathrm{Sq}^1x''_5=\mathrm{Sq}^1\mathrm{Sq}^2\beta_{(2,4)}x''_2=\mathrm{Sq}^3\beta_{(2,4)}x''_2=(\beta_{(2,4)}x''_2)^2=x''^2_3$. We have used the properties of Bockstein homomorphisms, (2.50) and the Adem relations (2.67).

Also note that

$$\begin{aligned}
\beta_{(2,8)}\mathcal{P}_2(x''_2) &= \frac{1}{8}\delta\mathcal{P}_2(x''_2)\mod 2\\
&= \frac{1}{8}\delta(x''_2\cup x''_2+x''_2\underset{1}{\cup}\delta x''_2)\\
&= \frac{1}{8}(\delta x''_2\cup x''_2+x''_2\cup\delta x''_2+\delta(x''_2\underset{1}{\cup}\delta x''_2))\\
&= \frac{1}{8}(2x''_2\cup\delta x''_2+\delta x''_2\underset{1}{\cup}\delta x''_2)\\
&= x''_2\cup(\frac{1}{4}\delta x''_2)+2(\frac{1}{4}\delta x''_2)\underset{1}{\cup}(\frac{1}{4}\delta x''_2)\\
&= x''_2\beta_{(2,4)}x''_2+2\beta_{(2,4)}x''_2\underset{1}{\cup}\beta_{(2,4)}x''_2\\
&= x''_2\beta_{(2,4)}x''_2+2\mathrm{Sq}^2\beta_{(2,4)}x''_2\\
&= \tilde{x}''_2x''_3+2x''_5\\
&= \tilde{x}''_2x''_3
\end{aligned} \tag{6.7}$$

We have used $\beta_{(2,8)}=\frac{1}{8}\delta\mod 2$, the Steenrod's formula (2.12), $\beta_{(2,4)}=\frac{1}{4}\delta\mod 2$, and the definition $\mathrm{Sq}^kx_n=x_n\underset{n-k}{\cup}x_n$.

There is a differential d_n corresponding to the Bockstein homomorphism $\beta_{(2,2^n)}:\mathrm{H}^*(-,\mathbb{Z}_{2^n})\to\mathrm{H}^{*+1}(-,\mathbb{Z}_2)$ associated to $0\to\mathbb{Z}_2\to\mathbb{Z}_{2^{n+1}}\to\mathbb{Z}_{2^n}\to 0$ [57]. See 2.5 for the definition of Bockstein homomorphisms.

So there are differentials such that $d_2(x''_3)=\tilde{x}''_2h_0^2$, $d_3(\tilde{x}''_2x''_3)=\tilde{x}''^2_2h_0^3$.

The E_2 page is shown in Figure 67.

Hence we have the following theorem

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_4)$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}_4$
3	0
4	$\mathbb{Z}\times\mathbb{Z}_8$
5	$\mathbb{Z}_2^2$

Theorem 114.

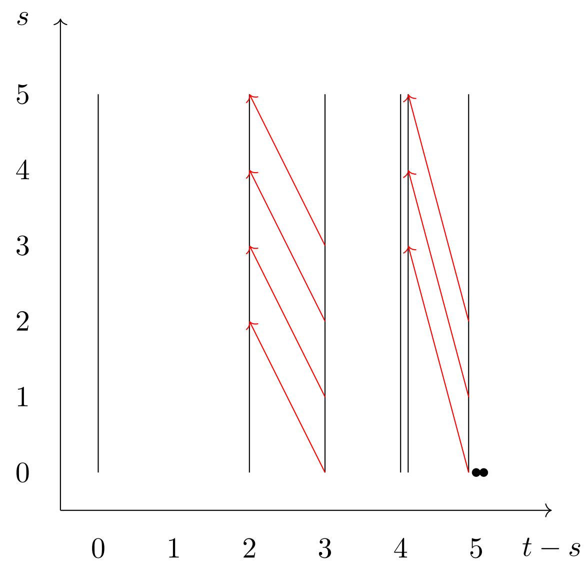


Figure 67: $\Omega_*^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_4)$

The 2d bordism invariant is x_2''.

The 4d bordism invariants are $\sigma, \mathcal{P}_2(x_2'')$.

The 5d bordism invariants are w_2w_3, x_5''.

6.3 B**O**(3)

6.3.1 $\Omega_d^{\mathbf{O}}(\mathrm{B}\mathbf{O}(3))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{BO}(3)) \tag{6.8}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)=\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^* \tag{6.9}$$

where $y_2^*=w_1^2$, $(y_2^2)^*=w_2^2$, $y_4^*=w_1^4$, $y_5^*=w_2w_3$, etc.

$$\mathrm{H}^*(\mathrm{BO}(3),\mathbb{Z}_2)=\mathbb{Z}_2[w_1',w_2',w_3'] \tag{6.10}$$

$$\begin{aligned}\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(3),\mathbb{Z}_2) &= \mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^*\otimes\mathbb{Z}_2[w_1',w_2',w_3'] \\ &= \mathcal{A}_2\oplus\Sigma\mathcal{A}_2\oplus 3\Sigma^2\mathcal{A}_2\oplus 4\Sigma^3\mathcal{A}_2\oplus 8\Sigma^4\mathcal{A}_2\oplus 11\Sigma^5\mathcal{A}_2\oplus\cdots\end{aligned} \tag{6.11}$$

i	$\Omega_i^{\mathrm{O}}(\mathrm{BO}(3))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^8$
5	$\mathbb{Z}_2^{11}$

Hence we have the following theorem

Theorem 115.

The 2d bordism invariants are $w_1^2, w_1'^2, w_2'$.

The 3d bordism invariant are $w_1'w_1^2, w_1'^3, w_1'w_2', w_3'$.

The 4d bordism invariants are $w_1^4, w_2^2, w_1^2w_1'^2, w_1^2w_2', w_1'w_3', w_1'^2w_2', w_1'^4, w_2'^2$.

The 5d bordism invariants are $w_2w_3, w_2^2w_1', w_1^4w_1', w_1^2w_1'^3, w_1^2w_1'w_2', w_1^2w_3', w_2'w_3', w_1'w_2'^2, w_1'^2w_3', w_1'^3w_2', w_1'^5$.

6.3.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbf{O}(3))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(3),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{BO}(3)) \tag{6.12}$$

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)=\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^5\mathcal{A}_2\oplus\cdots \tag{6.13}$$

$$\mathrm{H}^*(\mathrm{BO}(3),\mathbb{Z}_2)=\mathbb{Z}_2[w_1',w_2',w_3'] \tag{6.14}$$

where $\mathrm{Sq}^1w_2'=w_1'w_2'+w_3'$, $\mathrm{Sq}^1w_3'=w_1'w_3'$.

The E_2 page is shown in Figure 68.

Hence we have the following theorem

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BO}(3))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}^2\times\mathbb{Z}_2$
5	$\mathbb{Z}_2^6$

Theorem 116.

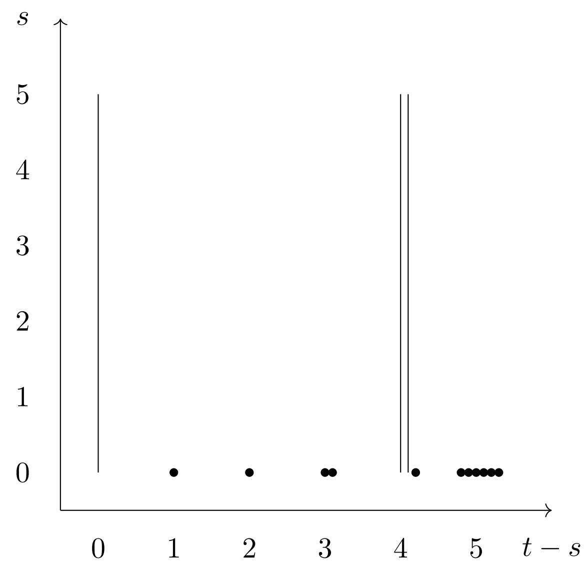


Figure 68: $\Omega_*^{\text{SO}}(\text{BO}(3))$

The 2d bordism invariant is w_2'.

The 3d bordism invariants are $w_1'^3, w_1'w_2' = w_3'$.

The 4d bordism invariants are $\sigma, p_1', w_1'^2 w_2'$.

The 5d bordism invariants are $w_2w_3, w_2^2 w_1', w_2'w_3', w_1'w_2'^2, w_1'^2 w_3' = w_1'^3 w_2', w_1'^5$.

6.4 B$\mathbf{O}$(4)

6.4.1 $\Omega_d^{\mathbf{O}}(\text{B}\mathbf{O}(4))$

$$\text{Ext}_{\mathcal{A}_2}^{s,t}(\text{H}^*(M\text{O},\mathbb{Z}_2)\otimes\text{H}^*(\text{BO}(4),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\text{O}}(\text{BO}(4)) \tag{6.15}$$

$$\text{H}^*(M\text{O},\mathbb{Z}_2)=\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^* \tag{6.16}$$

where $y_2^*=w_1^2$, $(y_2^2)^*=w_2^2$, $y_4^*=w_1^4$, $y_5^*=w_2w_3$, etc.

$$\text{H}^*(\text{BO}(4),\mathbb{Z}_2)=\mathbb{Z}_2[w_1',w_2',w_3',w_4'] \tag{6.17}$$

$$\begin{aligned}\text{H}^*(M\text{O},\mathbb{Z}_2)\otimes\text{H}^*(\text{BO}(4),\mathbb{Z}_2) &= \mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^*\otimes\mathbb{Z}_2[w_1',w_2',w_3',w_4'] \\ &= \mathcal{A}_2\oplus\Sigma\mathcal{A}_2\oplus 3\Sigma^2\mathcal{A}_2\oplus 4\Sigma^3\mathcal{A}_2\oplus 9\Sigma^4\mathcal{A}_2\oplus 12\Sigma^5\mathcal{A}_2\oplus\cdots\end{aligned} \tag{6.18}$$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{O}}(\mathrm{BO}(4))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^9$
5	$\mathbb{Z}_2^{12}$

Theorem 117.

The 2d bordism invariants are $w_1^2, w_1'^2, w_2'$.

The 3d bordism invariant are $w_1'w_1^2, w_1'^3, w_1'w_2', w_3'$.

The 4d bordism invariants are $w_1^4, w_2^2, w_1^2w_1'^2, w_1^2w_2', w_1'w_3', w_1'^2w_2', w_1'^4, w_2'^2, w_4'$.

The 5d bordism invariants are

$$w_2w_3, w_2^2w_1', w_1^4w_1', w_1^2w_1'^3, w_1^2w_1'w_2', w_1^2w_3', w_2'w_3', w_1'w_2'^2, w_1'^2w_3', w_1'^3w_2', w_1'^5, w_1'w_4'.$$

6.4.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbf{O}(4))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(4),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{BO}(4)) \tag{6.19}$$

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)=\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^5\mathcal{A}_2\oplus\cdots \tag{6.20}$$

$$\mathrm{H}^*(\mathrm{BO}(4),\mathbb{Z}_2)=\mathbb{Z}_2[w_1',w_2',w_3',w_4'] \tag{6.21}$$

where $\mathrm{Sq}^1w_2'=w_1'w_2'+w_3'$, $\mathrm{Sq}^1w_3'=w_1'w_3'$, $\mathrm{Sq}^1w_4'=w_1'w_4'$.

The E_2 page is shown in Figure 69.

Hence we have the following theorem

Theorem 118.

The 2d bordism invariant is w_2'.

The 3d bordism invariants are $w_1'^3, w_1'w_2'=w_3'$.

The 4d bordism invariants are $\sigma, p_1', w_1'^2w_2', w_4'$.

The 5d bordism invariants are $w_2w_3, w_2^2w_1', w_2'w_3', w_1'w_2'^2, w_1'^2w_3'=w_1'^3w_2', w_1'^5$.

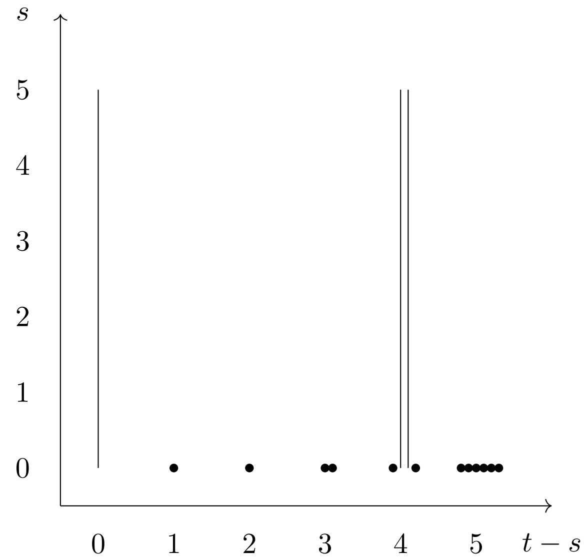


Figure 69: $\Omega_*^{\mathrm{SO}}(\mathrm{BO}(4))$

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BO}(4))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}^2 \times \mathbb{Z}_2^2$
5	$\mathbb{Z}_2^6$

6.5 B**O**(5)

6.5.1 $\Omega_d^{\mathbf{O}}(\mathrm{B}\mathbf{O}(5))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(5),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{BO}(5)) \tag{6.22}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)=\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^* \tag{6.23}$$

where $y_2^*=w_1^2$, $(y_2^2)^*=w_2^2$, $y_4^*=w_1^4$, $y_5^*=w_2w_3$, etc.

$$\mathrm{H}^*(\mathrm{BO}(5),\mathbb{Z}_2)=\mathbb{Z}_2[w_1',w_2',w_3',w_4',w_5'] \tag{6.24}$$

$$\begin{aligned}\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(5),\mathbb{Z}_2) &= \mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^*\otimes\mathbb{Z}_2[w_1',w_2',w_3',w_4',w_5'] \\ &= \mathcal{A}_2\oplus\Sigma\mathcal{A}_2\oplus 3\Sigma^2\mathcal{A}_2\oplus 4\Sigma^3\mathcal{A}_2\oplus 9\Sigma^4\mathcal{A}_2\oplus 13\Sigma^5\mathcal{A}_2\oplus\cdots\end{aligned} \tag{6.25}$$

Hence we have the following theorem

i	$\Omega_i^{\mathrm{O}}(\mathrm{BO}(5))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^9$
5	$\mathbb{Z}_2^{13}$

Theorem 119.

The 2d bordism invariants are $w_1^2, w_1'^2, w_2'$.

The 3d bordism invariant are $w_1'w_1^2, w_1'^3, w_1'w_2', w_3'$.

The 4d bordism invariants are $w_1^4, w_2^2, w_1^2w_1'^2, w_1^2w_2', w_1'w_3', w_1'^2w_2', w_1'^4, w_2'^2, w_4'$.

The 5d bordism invariants are

$$w_2w_3, w_2^2w_1', w_1^4w_1', w_1^2w_1'^3, w_1^2w_1'w_2', w_1^2w_3', w_2'w_3', w_1'w_2'^2, w_1'^2w_3', w_1'^3w_2', w_1'^5, w_1'w_4', w_5'.$$

6.5.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbf{O}(5))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{BO}(5),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{BO}(5)) \tag{6.26}$$

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)=\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^5\mathcal{A}_2\oplus\cdots \tag{6.27}$$

$$\mathrm{H}^*(\mathrm{BO}(5),\mathbb{Z}_2)=\mathbb{Z}_2[w_1',w_2',w_3',w_4',w_5'] \tag{6.28}$$

where $\mathrm{Sq}^1w_2'=w_1'w_2'+w_3'$, $\mathrm{Sq}^1w_3'=w_1'w_3'$, $\mathrm{Sq}^1w_4'=w_1'w_4'+w_5'$.

The E_2 page is shown in Figure 70.

Hence we have the following theorem

Theorem 120.

The 2d bordism invariant is w_2'.

The 3d bordism invariants are $w_1'^3, w_1'w_2'=w_3'$.

The 4d bordism invariants are $\sigma, p_1', w_1'^2w_2', w_4'$.

The 5d bordism invariants are $w_2w_3, w_2^2w_1', w_2'w_3', w_1'w_2'^2, w_1'^2w_3'=w_1'^3w_2', w_1'^5, w_1'w_4'=w_5'$.

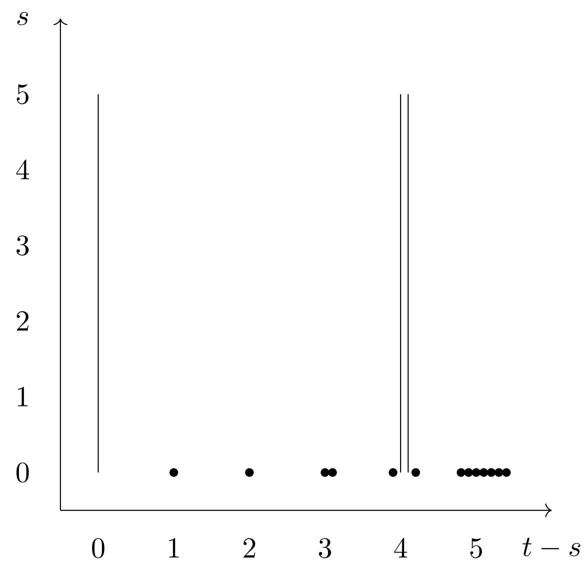


Figure 70: $\Omega_*^{\mathrm{SO}}(\mathrm{BO}(5))$

i	$\Omega_i^{\mathrm{SO}}(\mathrm{BO}(5))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}^2 \times \mathbb{Z}_2^2$
5	$\mathbb{Z}_2^7$

6.6 $\mathrm{B}\mathbb{Z}_{2n} \times \mathrm{B}^2\mathbb{Z}_n$

6.6.1 $\Omega_d^{\mathbf{O}}(\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2)$

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4) = \mathbb{Z}_4[a, b]/(a^2 = 2b) \tag{6.29}$$

where $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_4)$.

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_2) = \Lambda_{\mathbb{Z}_2}(\tilde{a}) \otimes \mathbb{Z}_2[\tilde{b}] \tag{6.30}$$

where $\tilde{a} = a \mod 2 \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_2)$, $\tilde{b} = b \mod 2 \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_4, \mathbb{Z}_2)$.

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots] \tag{6.31}$$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2) \tag{6.32}$$

$$\mathrm{H}^*(MO,\mathbb{Z}_2)=\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^* \tag{6.33}$$

where $y_2^*=w_1^2$, $(y_2^2)^*=w_2^2$, $y_4^*=w_1^4$, $y_5^*=w_2w_3$, etc.

$$\begin{aligned}&\mathrm{H}^*(MO,\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_4\times\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2)\\ =\;&\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^*\otimes\Lambda_{\mathbb{Z}_2}(\tilde a)\otimes\mathbb{Z}_2[\tilde b]\otimes\mathbb{Z}_2[\tilde x_2,x_3,x_5,x_9,\dots]\\ =\;&\mathcal{A}_2\oplus\Sigma\mathcal{A}_2\oplus3\Sigma^2\mathcal{A}_2\oplus4\Sigma^3\mathcal{A}_2\oplus8\Sigma^4\mathcal{A}_2\oplus12\Sigma^5\mathcal{A}_2\oplus\cdots\end{aligned} \tag{6.34}$$

Hence we have the following theorem

Theorem 121. The bordism groups are

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_4\times\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^8$
5	$\mathbb{Z}_2^{12}$

The 2d bordism invariants are $\tilde b, x_2, w_1^2$.

The 3d bordism invariants are $\tilde a\tilde b, x_3, \tilde a x_2, \tilde a w_1^2$.

The 4d bordism invariants are $\tilde a x_3, \tilde b x_2, \tilde b^2, x_2^2, w_1^4, w_2^2, \tilde b w_1^2, x_2w_1^2$.

The 5d bordism invariants are $\tilde a x_2^2, \tilde b x_3, x_2x_3, \tilde a\tilde b^2, x_5, \tilde a\tilde b x_2, w_2w_3, \tilde a w_2^2, \tilde a w_1^4, \tilde a\tilde b w_1^2, x_3w_1^2= w_1^3x_2, \tilde a x_2 w_1^2$.

6.6.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_4\times\mathrm{B}^2\mathbb{Z}_2)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_4\times\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_4\times\mathrm{B}^2\mathbb{Z}_2) \tag{6.35}$$

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)=\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^5\mathcal{A}_2\oplus\cdots \tag{6.36}$$

Note that $\beta_{(2,4)}a=\tilde b$, $\mathrm{Sq}^1x_2=x_3$, $\mathrm{Sq}^1(\tilde a x_2)=\tilde a x_3$, $\mathrm{Sq}^1(\tilde b x_2)=\tilde b x_3$, $\beta_{(2,4)}(ab)=\tilde b^2$, $\beta_{(2,4)}(\mathcal{P}_2(x_2))=x_2x_3+x_5$, $\mathrm{Sq}^1(x_2x_3)=\mathrm{Sq}^1x_5=x_3^2$, $\mathrm{Sq}^1(\tilde a\tilde b x_2)=\tilde a\tilde b x_3$, $\beta_{(2,4)}(a\mathcal{P}_2(x_2))=\tilde b x_2^2+\tilde a(x_2x_3+x_5)$, $\beta_{(2,4)}(ab^2)=\tilde b^3$, $\beta_{(2,4)}(a(\sigma \mod 4))=\tilde b w_2^2$.

There is a differential d_2 corresponding to the Bockstein homomorphism $\beta_{(2,4)}: \mathrm{H}^*(-,\mathbb{Z}_4) \to \mathrm{H}^{*+1}(-,\mathbb{Z}_2)$ associated to $0 \to \mathbb{Z}_2 \to \mathbb{Z}_8 \to \mathbb{Z}_4 \to 0$ [57]. See 2.5 for the definition of Bockstein homomorphisms.

So there are differentials such that $d_2(\tilde{b}) = \tilde{a}h_0^2$, $d_2(\tilde{b}^2) = \tilde{a}\tilde{b}h_0^2$, $d_2(x_2x_3 + x_5) = x_2^2h_0^2$, $d_2(\tilde{b}x_2^2 + \tilde{a}(x_2x_3 + x_5)) = \tilde{a}x_2^2h_0^2$, $d_2(\tilde{b}^3) = \tilde{a}\tilde{b}^2h_0^2$, $d_2(\tilde{b}w_2^2) = \tilde{a}w_2^2h_0^2$.

The E_2 page is shown in Figure 71.

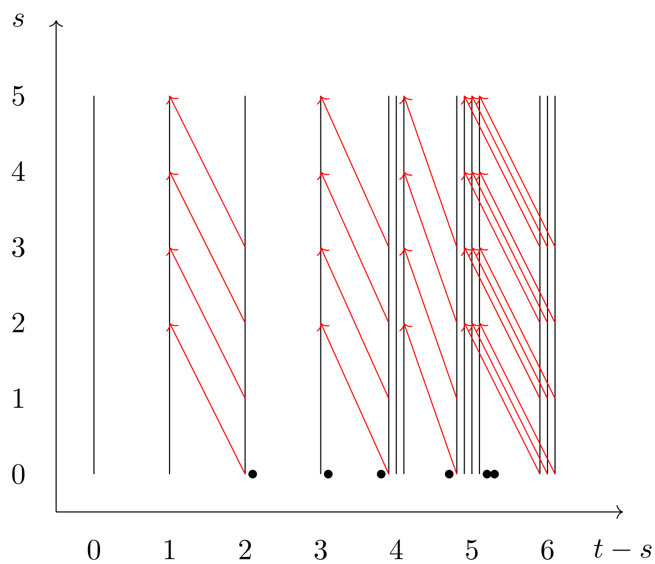


Figure 71: $\Omega_*^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2)$

Hence we have the following theorem

Theorem 122. The bordism groups are

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_4$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_4 \times \mathbb{Z}_2$
4	$\mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_2$
5	$\mathbb{Z}_4^3 \times \mathbb{Z}_2^3$

The 2d bordism invariant is x_2.

The 3d bordism invariants are ab and $\tilde{a}x_2$.

The 4d bordism invariants are σ, $\mathcal{P}_2(x_2)$ and $\tilde{b}x_2$.

The 5d bordism invariants are $a\mathcal{P}_2(x_2)$, ab^2, $a(\sigma \mod 4)$, $x_5 = x_2x_3$, $\tilde{a}\tilde{b}x_2$ and w_2w_3.

6.6.3 $\Omega_d^{\mathbf{O}}(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2[a] \tag{6.37}$$

where $a\in\mathrm{H}^1(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_3)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge \tag{6.38}$$

Since $\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_3)=0$, we have $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge=0$.

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge \tag{6.39}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)=\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^* \tag{6.40}$$

where $y_2^*=w_1^2$, $(y_2^2)^*=w_2^2$, $y_4^*=w_1^4$, $y_5^*=w_2w_3$, etc.

$$\begin{aligned}\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2) &= \mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^*\otimes\mathbb{Z}_2[a] \\ &= \mathcal{A}_2\oplus\Sigma\mathcal{A}_2\oplus2\Sigma^2\mathcal{A}_2\oplus2\Sigma^3\mathcal{A}_2\oplus4\Sigma^4\mathcal{A}_2\oplus5\Sigma^5\mathcal{A}_2\oplus\cdots\end{aligned} \tag{6.41}$$

Hence we have the following theorem

Theorem 123. The bordism groups are

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_6\times\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^5$

The 2d bordism invariants are w_1^2,a^2.

The 3d bordism invariants are a^3,aw_1^2.

The 4d bordism invariants are $a^4,a^2w_1^2,w_1^4,w_2^2$.

The 5d bordism invariants are $a^5,a^3w_1^2,aw_1^4,aw_2^2,w_2w_3$.

6.6.4 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO} \wedge (\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{6.42}$$

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_2) = \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$, we have $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge = \Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2)$.

The E_2 page is shown in Figure 72.

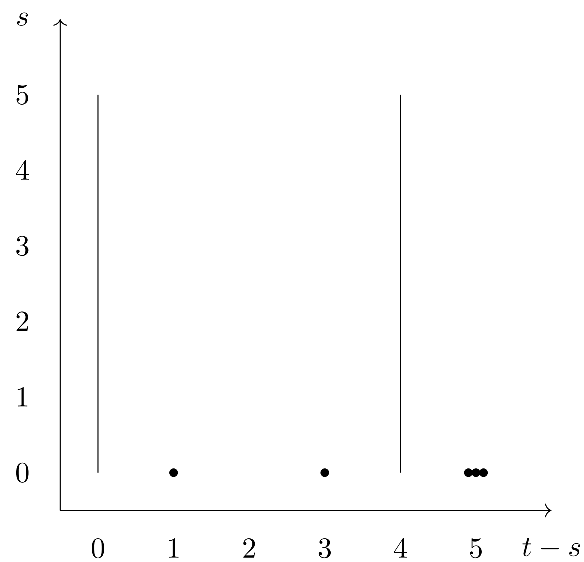


Figure 72: $\Omega_*^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2)$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO} \wedge (\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)_+, \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{6.43}$$

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3) = \mathrm{H}^*(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3)$, we have $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge = \Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_3 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge$.

Hence we have the following theorem

Theorem 124. The bordism groups are

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_6 \times \mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_3 \times \mathbb{Z}_2$
2	$\mathbb{Z}_3$
3	$\mathbb{Z}_3^2 \times \mathbb{Z}_2$
4	$\mathbb{Z} \times \mathbb{Z}_3^2$
5	$\mathbb{Z}_2^3 \times \mathbb{Z}_3^2 \times \mathbb{Z}_9$

The 2d bordism invariant is x_2'.

The 3d bordism invariants are $a'b', a'x_2', a^3$.

The 4d bordism invariants are σ, $a'x_3'(=b'x_2')$ and $x_2'^2$.

The 5d bordism invariants are $a^5, aw_2^2, w_2w_3, a'b'x_2', a'x_2'^2, \mathfrak{P}_3(b')$.

Here $\mathfrak{P}_3$ is the Postnikov square.

6.7 $\mathrm{B}\mathbb{Z}_{2n^2} \times \mathrm{B}^2\mathbb{Z}_n$

6.7.1 $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2)$

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8) = \mathbb{Z}_8[a,b]/(a^2 = 4b) \tag{6.44}$$

where $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$, $b \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_8)$.

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_2) = \Lambda_{\mathbb{Z}_2}(\tilde{a}) \otimes \mathbb{Z}_2[\tilde{b}] \tag{6.45}$$

where $\tilde{a} = a \mod 2 \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_2)$, $\tilde{b} = b \mod 2 \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_8, \mathbb{Z}_2)$.

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, x_5, x_9, \dots] \tag{6.46}$$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2) \tag{6.47}$$

$$\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) = \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \tag{6.48}$$

where $y_2^* = w_1^2$, $(y_2^2)^* = w_2^2$, $y_4^* = w_1^4$, $y_5^* = w_2w_3$, etc.

$$\begin{aligned} & \mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_4 \times \mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2) \\ = \; & \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \otimes \Lambda_{\mathbb{Z}_2}(\tilde{a}) \otimes \mathbb{Z}_2[\tilde{b}] \otimes \mathbb{Z}_2[\tilde{x}_2, x_3, x_5, x_9, \dots] \\ = \; & \mathcal{A}_2 \oplus \Sigma\mathcal{A}_2 \oplus 3\Sigma^2\mathcal{A}_2 \oplus 4\Sigma^3\mathcal{A}_2 \oplus 8\Sigma^4\mathcal{A}_2 \oplus 12\Sigma^5\mathcal{A}_2 \oplus \cdots \end{aligned} \tag{6.49}$$

Hence we have the following theorem

Theorem 125. The bordism groups are

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_8\times\mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$
4	$\mathbb{Z}_2^8$
5	$\mathbb{Z}_2^{12}$

The 2d bordism invariants are $\tilde{b}, x_2, w_1^2$.

The 3d bordism invariants are $\tilde{a}\tilde{b}, x_3, \tilde{a}x_2, \tilde{a}w_1^2$.

The 4d bordism invariants are $\tilde{a}x_3, \tilde{b}x_2, \tilde{b}^2, x_2^2, w_1^4, w_2^2, \tilde{b}w_1^2, x_2w_1^2$.

The 5d bordism invariants are $\tilde{a}x_2^2, \tilde{b}x_3, x_2x_3, \tilde{a}\tilde{b}^2, x_5, \tilde{a}\tilde{b}x_2, w_2w_3, \tilde{a}w_2^2, \tilde{a}w_1^4, \tilde{a}\tilde{b}w_1^2, x_3w_1^2= w_1^3x_2, \tilde{a}x_2w_1^2$.

6.7.2 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_8\times\mathrm{B}^2\mathbb{Z}_2)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_8\times\mathrm{B}^2\mathbb{Z}_2,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_8\times\mathrm{B}^2\mathbb{Z}_2) \tag{6.50}$$

$$\mathrm{H}^*(M\mathrm{SO},\mathbb{Z}_2)=\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^4\mathcal{A}_2/\mathcal{A}_2\mathrm{Sq}^1\oplus\Sigma^5\mathcal{A}_2\oplus\cdots \tag{6.51}$$

Note that $\beta_{(2,8)}a = \tilde{b}$, $\mathrm{Sq}^1x_2 = x_3$, $\mathrm{Sq}^1(\tilde{a}x_2) = \tilde{a}x_3$, $\mathrm{Sq}^1(\tilde{b}x_2) = \tilde{b}x_3$, $\beta_{(2,8)}(ab) = \tilde{b}^2$, $\beta_{(2,4)}(\mathcal{P}_2(x_2)) = x_2x_3+x_5$, $\mathrm{Sq}^1(x_2x_3)=\mathrm{Sq}^1x_5=x_3^2$, $\mathrm{Sq}^1(\tilde{a}\tilde{b}x_2)=\tilde{a}\tilde{b}x_3$, $\beta_{(2,4)}((a \mod 4)\mathcal{P}_2(x_2)) = 2\beta_{(2,8)}(a)x_2^2+\tilde{a}(x_2x_3+x_5) = 2\tilde{b}x_2^2+\tilde{a}(x_2x_3+x_5) = \tilde{a}(x_2x_3+x_5)$, $\beta_{(2,8)}(ab^2) = \tilde{b}^3$, $\beta_{(2,8)}(a(\sigma \mod 8)) = \tilde{b}w_2^2$.

There is a differential d_n corresponding to the Bockstein homomorphism $\beta_{(2,2^n)}:\mathrm{H}^*(-,\mathbb{Z}_{2^n})\to\mathrm{H}^{*+1}(-,\mathbb{Z}_2)$ associated to $0\to\mathbb{Z}_2\to\mathbb{Z}_{2^{n+1}}\to\mathbb{Z}_{2^n}\to 0$ [57]. See 2.5 for the definition of Bockstein homomorphisms.

So there are differentials such that $d_3(\tilde{b})=\tilde{a}h_0^2$, $d_3(\tilde{b}^2)=\tilde{a}\tilde{b}h_0^2$, $d_2(x_2x_3+x_5)=x_2^2h_0^2$, $d_2(\tilde{a}(x_2x_3+x_5))=\tilde{a}x_2^2h_0^2$, $d_3(\tilde{b}^3)=\tilde{a}\tilde{b}^2h_0^2$, $d_3(\tilde{b}w_2^2)=\tilde{a}w_2^2h_0^2$.

The E_2 page is shown in Figure 73.

Hence we have the following theorem

Theorem 126. The bordism groups are

The 2d bordism invariant is x_2.

The 3d bordism invariants are ab and $\tilde{a}x_2$.

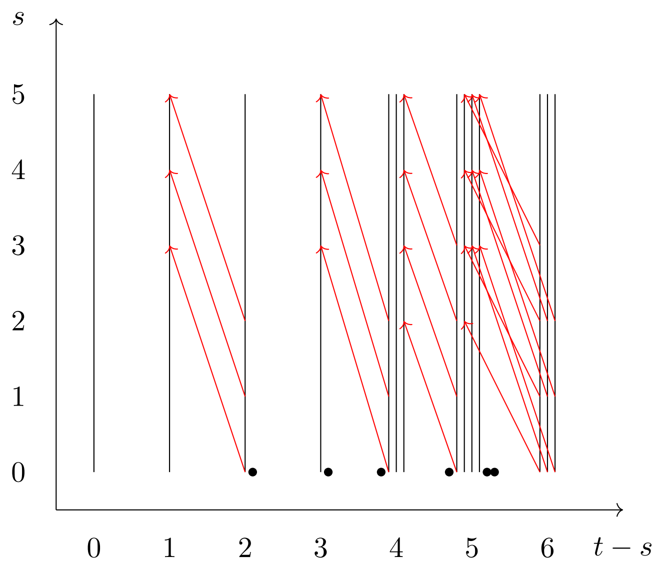


Figure 73: $\Omega_*^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2)$

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_8 \times \mathrm{B}^2\mathbb{Z}_2)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_8$
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_8 \times \mathbb{Z}_2$
4	$\mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_2$
5	$\mathbb{Z}_4 \times \mathbb{Z}_8^2 \times \mathbb{Z}_2^3$

The 4d bordism invariants are σ, $\mathcal{P}_2(x_2)$ and $\tilde{b}x_2$.

The 5d bordism invariants are $(a \mod 4)\mathcal{P}_2(x_2)$, ab^2, $a(\sigma \mod 8)$, $x_5 = x_2x_3$, $\tilde{a}\tilde{b}x_2$ and w_2w_3.

6.7.3 $\Omega_d^{\mathbf{O}}(\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_2) = \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[a] \tag{6.52}$$

where $a \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_3) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge \tag{6.53}$$

Since $\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_3) = 0$, we have $\Omega_d^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge = 0$.

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_{18} \times \mathrm{B}^2\mathbb{Z}_3)_2^\wedge \tag{6.54}$$

$$\mathrm{H}^*(MO,\mathbb{Z}_2)=\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^* \tag{6.55}$$

where $y_2^*=w_1^2$, $(y_2^2)^*=w_2^2$, $y_4^*=w_1^4$, $y_5^*=w_2w_3$, etc.

$$\begin{aligned}\mathrm{H}^*(MO,\mathbb{Z}_2)\otimes\mathrm{H}^*(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2) &= \mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^*\otimes\mathbb{Z}_2[a] \\ &= \mathcal{A}_2\oplus\Sigma\mathcal{A}_2\oplus2\Sigma^2\mathcal{A}_2\oplus2\Sigma^3\mathcal{A}_2\oplus4\Sigma^4\mathcal{A}_2\oplus5\Sigma^5\mathcal{A}_2\oplus\cdots\end{aligned} \tag{6.56}$$

Hence we have the following theorem

Theorem 127. The bordism groups are

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$
4	$\mathbb{Z}_2^4$
5	$\mathbb{Z}_2^5$

The 2d bordism invariants are w_1^2, a^2.

The 3d bordism invariants are a^3, aw_1^2.

The 4d bordism invariants are $a^4, a^2w_1^2, w_1^4, w_2^2$.

The 5d bordism invariants are $a^5, a^3w_1^2, aw_1^4, aw_2^2, w_2w_3$.

6.7.4 $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge. \tag{6.57}$$

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_2)=\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)$, we have $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_2^\wedge=\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_2)$.

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO}\wedge(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_+,\mathbb{Z}_3),\mathbb{Z}_3)\Rightarrow\Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge. \tag{6.58}$$

Since $\mathrm{H}^*(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_3)=\mathrm{H}^*(\mathrm{B}\mathbb{Z}_9\times\mathrm{B}^2\mathbb{Z}_3,\mathbb{Z}_3)$, we have $\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_{18}\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge=\Omega_d^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_9\times\mathrm{B}^2\mathbb{Z}_3)_3^\wedge$.

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_9,\mathbb{Z}_9)=\Lambda_{\mathbb{Z}_9}(a')\otimes\mathbb{Z}_9[b']. \tag{6.59}$$

where $a' \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_9, \mathbb{Z}_9)$, $b' \in \mathrm{H}^2(\mathrm{B}\mathbb{Z}_9, \mathbb{Z}_9)$.

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_9, \mathbb{Z}_3) = \Lambda_{\mathbb{Z}_3}(\tilde{a}') \otimes \mathbb{Z}_3[\tilde{b}']. \tag{6.60}$$

where $\tilde{a}' = a' \mod 3$, $\tilde{b}' = b' \mod 3$, $\tilde{b}' = \beta_{(3,9)}(a')$.

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3[x_2', x_8', \dots] \otimes \Lambda_{\mathbb{Z}_3}(x_3', x_7', \dots) \tag{6.61}$$

Note that $\beta_{(3,3)}(\tilde{a}') = 3\beta_{(3,9)}(a') = 3\tilde{b}' = 0$, $\beta_{(3,3)}(x_2') = x_3'$, $\beta_{(3,3)}(x_2'^2) = 2x_2'x_3'$, $\beta_{(3,9)}(a'b') = \tilde{b}'^2$, $\beta_{(3,9)}(a'b'^2) = \tilde{b}'^3$, $\beta_{(3,3)}(\tilde{a}'x_2') = \tilde{a}'x_3'$, $\beta_{(3,3)}(\tilde{b}'x_2') = \tilde{b}'x_3'$, $\beta_{(3,3)}(\tilde{a}'\tilde{b}'x_2') = \tilde{a}'\tilde{b}'x_3'$, $\beta_{(3,3)}(\tilde{a}'x_2'^2) = 2\tilde{a}'x_2'x_3'$.

There is a differential d_2 corresponding to the $(3,9)$-Bockstein [57].

So there are differentials such that $d_2(\tilde{b}') = \tilde{a}'h_0'^2$, $d_2(\tilde{b}'^2) = \tilde{a}'\tilde{b}'h_0'^2$, $d_2(\tilde{b}'^3) = \tilde{a}'\tilde{b}'^2h_0'^2$.

The E_2 page is shown in Figure 74.

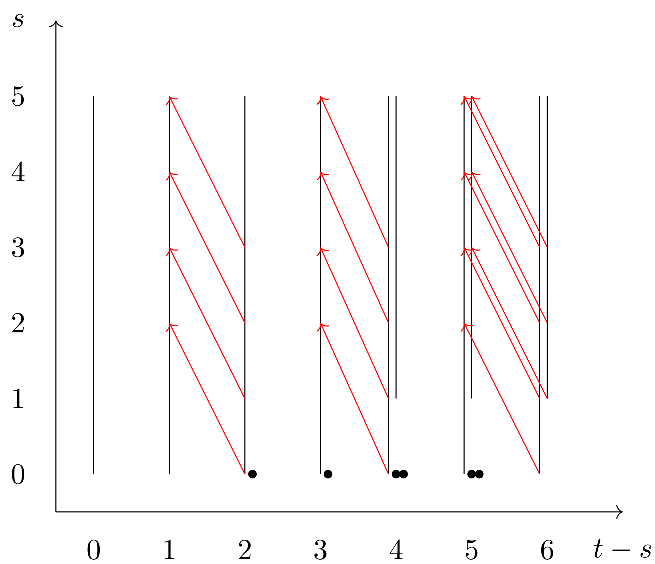


Figure 74: $\Omega_*^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_9 \times \mathrm{B}^2\mathbb{Z}_3)_3^\wedge$

Hence we have the following theorem

Theorem 128. The bordism groups are

The 2d bordism invariant is x_2'.

The 3d bordism invariants are $a'b', \tilde{a}'x_2', a^3$.

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_{18}\times \mathrm{B}^2\mathbb{Z}_3)$
0	$\mathbb{Z}$
1	$\mathbb{Z}_9\times\mathbb{Z}_2$
2	$\mathbb{Z}_3$
3	$\mathbb{Z}_9\times\mathbb{Z}_3\times\mathbb{Z}_2$
4	$\mathbb{Z}\times\mathbb{Z}_3^2$
5	$\mathbb{Z}_2^3\times\mathbb{Z}_3^3\times\mathbb{Z}_{27}$

The 4d bordism invariants are $\sigma, \tilde{b}'x_2', x_2'^2$.

The 5d bordism invariants are $a^5, aw_2^2, w_2w_3, \tilde{a}'(\sigma \mod 3), \tilde{a}'\tilde{b}'x_2', \tilde{a}'x_2'^2, \mathfrak{P}_3(b')$.

Here $\mathfrak{P}_3$ is the Postnikov square.

6.8 $\mathrm{B}(\mathbb{Z}_2\ltimes \mathrm{PSU}(N))$

For $N>2$, the outer automorphism group of $\mathrm{PSU}(N)$ is $\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathrm{PSU}(N)$ via complex conjugation.

6.8.1 $\Omega_3^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2\ltimes \mathrm{PSU}(3)))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2\ltimes \mathrm{PSU}(3)),\mathbb{Z}_2),\mathbb{Z}_2)\Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2\ltimes \mathrm{PSU}(3))) \tag{6.62}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)=\mathcal{A}_2\otimes\mathbb{Z}_2[y_2,y_4,y_5,y_6,y_8,\dots]^* \tag{6.63}$$

where $y_2^*=w_1^2$, $(y_2^2)^*=w_2^2$, $y_4^*=w_1^4$, $y_5^*=w_2w_3$, etc.

We have a fibration

$$\mathrm{BPSU}(3)\to \mathrm{B}(\mathbb{Z}_2\ltimes \mathrm{PSU}(3))\to \mathrm{B}\mathbb{Z}_2. \tag{6.64}$$

and

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2[a] \tag{6.65}$$

$$\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_2)=\mathbb{Z}_2[c_2,c_3] \tag{6.66}$$

By Serre spectral sequence, we have

$$\mathrm{H}^p(\mathrm{B}\mathbb{Z}_2,\mathrm{H}^q(\mathrm{BPSU}(3),\mathbb{Z}_2))\Rightarrow \mathrm{H}^{p+q}(\mathrm{B}(\mathbb{Z}_2\ltimes \mathrm{PSU}(3)),\mathbb{Z}_2). \tag{6.67}$$

The relevant piece is shown in Figure 75.

3	0	0	0	0
2	0	0	0	0
1	0	0	0	0
0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	0	1	2	3

Figure 75: Serre spectral sequence for $(\mathrm{B}\mathbb{Z}_2, \mathrm{BPSU}(3))$ with coefficients $\mathbb{Z}_2$

Hence $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)), \mathbb{Z}_2) = \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$ for $* \leq 3$.

$$\begin{aligned} & \mathrm{H}^*(M\mathrm{O}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2) \\ = \ & \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \dots]^* \otimes \mathbb{Z}_2[a] \\ = \ & \mathcal{A}_2 \oplus \Sigma\mathcal{A}_2 \oplus 2\Sigma^2\mathcal{A}_2 \oplus 2\Sigma^3\mathcal{A}_2 \oplus \cdots \end{aligned} \tag{6.68}$$

Hence we have the following theorem

Theorem 129. The bordism groups are

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$
3	$\mathbb{Z}_2^2$

The 1d bordism invariant is a.

The 2d bordism invariants are a^2, w_1^2.

The 3d bordism invariants are a^3, aw_1^2.

6.8.2 $\Omega_3^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)))_2^\wedge \tag{6.69}$$

$$\mathrm{Ext}_{\mathcal{A}_3}^{s,t}(\mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_3) \otimes \mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)), \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(3)))_3^\wedge \tag{6.70}$$

$$\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_3)=(\mathbb{Z}_3[z_2,z_8,z_{12}]\otimes\Lambda_{\mathbb{Z}_3}(z_3,z_7))/(z_2z_3,z_2z_7,z_2z_8+z_3z_7) \tag{6.71}$$

By Serre spectral sequence, we have

$$\mathrm{H}^p(\mathrm{B}\mathbb{Z}_2,\mathrm{H}^q(\mathrm{BPSU}(3),\mathbb{Z}_3))\Rightarrow\mathrm{H}^{p+q}(\mathrm{B}(\mathbb{Z}_2\ltimes\mathrm{PSU}(3)),\mathbb{Z}_3). \tag{6.72}$$

The relevant piece is shown in Figure 76.

3	$\mathbb{Z}_3$	0	0	0
2	$\mathbb{Z}_3$	0	0	0
1	0	0	0	0
0	$\mathbb{Z}_3$	0	0	0
	0	1	2	3

Figure 76: Serre spectral sequence for $(\mathrm{B}\mathbb{Z}_2,\mathrm{BPSU}(3))$ with coefficients $\mathbb{Z}_3$

Hence $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2\ltimes\mathrm{PSU}(3)),\mathbb{Z}_3)=\mathrm{H}^*(\mathrm{BPSU}(3),\mathbb{Z}_3)$ for $*\le 3$.

Combining this with previous results, we have the following theorem

Theorem 130. The bordism groups are

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2\ltimes\mathrm{PSU}(3)))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_3$
3	$\mathbb{Z}_2$

The 1d bordism invariant is a.

The 2d bordism invariant is z_2.

The 3d bordism invariant is a^3.

Here $z_2=w_2(\mathrm{PSU}(3))\in\mathrm{H}^2(\mathrm{BPSU}(3),\mathbb{Z}_3)$ is the generalized Stiefel-Whitney class of the principal PSU(3) bundle.

6.8.3 $\Omega_3^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2)\otimes \mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)),\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4))) \tag{6.73}$$

$$\mathrm{H}^*(M\mathrm{O},\mathbb{Z}_2) = \mathcal{A}_2 \otimes \mathbb{Z}_2[y_2, y_4, y_5, y_6, y_8, \ldots]^* \tag{6.74}$$

where $y_2^* = w_1^2$, $(y_2^2)^* = w_2^2$, $y_4^* = w_1^4$, $y_5^* = w_2 w_3$, etc.

We have a fibration

$$\mathrm{BPSU}(4) \to \mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)) \to \mathrm{B}\mathbb{Z}_2. \tag{6.75}$$

and

$$\mathrm{H}^*(\mathrm{B}\mathbb{Z}_2,\mathbb{Z}_2) = \mathbb{Z}_2[a] \tag{6.76}$$

We also have a fibration

$$\mathrm{BSU}(4) \to \mathrm{BPSU}(4) \to \mathrm{B}^2\mathbb{Z}_4 \tag{6.77}$$

and

$$\mathrm{H}^*(\mathrm{BSU}(4),\mathbb{Z}_2) = \mathbb{Z}_2[c_2, c_3, c_4] \tag{6.78}$$

$$\mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_4,\mathbb{Z}_2) = \mathbb{Z}_2[\tilde{x}_2'', x_3'', x_5'', x_9'', \ldots] \tag{6.79}$$

By Serre spectral sequence, we have

$$\mathrm{H}^p(\mathrm{B}^2\mathbb{Z}_4, \mathrm{H}^q(\mathrm{BSU}(4),\mathbb{Z}_2)) \Rightarrow \mathrm{H}^{p+q}(\mathrm{BPSU}(4),\mathbb{Z}_2). \tag{6.80}$$

The relevant piece is shown in Figure 77.

Hence $\mathrm{H}^*(\mathrm{BPSU}(4),\mathbb{Z}_2) = \mathrm{H}^*(\mathrm{B}^2\mathbb{Z}_4,\mathbb{Z}_2)$ for $* \leq 3$.

Again by Serre spectral sequence, we have

$$\mathrm{H}^p(\mathrm{B}\mathbb{Z}_2, \mathrm{H}^q(\mathrm{BPSU}(4),\mathbb{Z}_2)) \Rightarrow \mathrm{H}^{p+q}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)),\mathbb{Z}_2). \tag{6.81}$$

The relevant piece is shown in Figure 78.

3	0	0	0	0
2	0	0	0	0
1	0	0	0	0
0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	0	1	2	3

Figure 77: Serre spectral sequence for $(\mathrm{B}^2\mathbb{Z}_4, \mathrm{BSU}(4))$

3	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	0	0	0	0	0
0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	0	1	2	3	4

Figure 78: Serre spectral sequence for $(\mathrm{B}\mathbb{Z}_2, \mathrm{BPSU}(4))$

There are no differentials,

$$\mathrm{H}^n(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)), \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & n=0 \\ \mathbb{Z}_2 & n=1 \\ \mathbb{Z}_2^2 & n=2 \\ \mathbb{Z}_2^3 & n=3 \end{cases} \tag{6.82}$$

$$\begin{aligned} & \mathrm{H}^*(MO, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)), \mathbb{Z}_2) \\ = \; & \mathcal{A}_2 \oplus \Sigma\mathcal{A}_2 \oplus 3\Sigma^2\mathcal{A}_2 \oplus 4\Sigma^3\mathcal{A}_2 \oplus \cdots \end{aligned} \tag{6.83}$$

Hence we have the following theorem

Theorem 131. The bordism groups are

The 1d bordism invariant is a.

i	$\Omega_i^{\mathrm{O}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^3$
3	$\mathbb{Z}_2^4$

The 2d bordism invariants are $a^2, \tilde{z}_2', w_1^2$.

The 3d bordism invariants are $a^3, z_3', a\tilde{z}_2', aw_1^2$.

Here $z_2' = w_2(\mathrm{PSU}(4)) \in \mathrm{H}^2(\mathrm{BPSU}(4), \mathbb{Z}_4)$ is the generalized Stiefel-Whitney class of the principal PSU(4) bundle, $\tilde{z}_2' = z_2' \mod 2$, $z_3' = \beta_{(2,4)} z_2'$.

6.8.4 $\Omega_3^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$

$$\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathrm{H}^*(M\mathrm{SO}, \mathbb{Z}_2) \otimes \mathrm{H}^*(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4))) \quad (6.84)$$

There is a differential d_2 corresponding to the Bockstein homomorphism $\beta_{(2,4)} : \mathrm{H}^*(-, \mathbb{Z}_4) \to \mathrm{H}^{*+1}(-, \mathbb{Z}_2)$ associated to $0 \to \mathbb{Z}_2 \to \mathbb{Z}_8 \to \mathbb{Z}_4 \to 0$ [57]. See 2.5 for the definition of Bockstein homomorphisms.

Since $\beta_{(2,4)}(z_2') = z_3'$, there is a differential such that $d_2(z_3') = \tilde{z}_2' h_0^2$.

The E_2 page is shown in Figure 79.

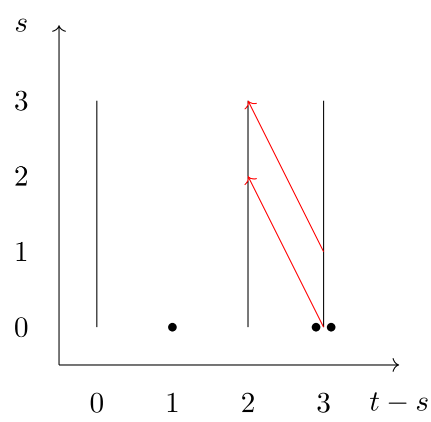


Figure 79: $\Omega_*^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$

Hence we have the following theorem

Theorem 132. The bordism groups are

The 1d bordism invariant is a.

i	$\Omega_i^{\mathrm{SO}}(\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(4)))$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_4$
3	$\mathbb{Z}_2^2$

The 2d bordism invariant is z_2'.

The 3d bordism invariants are $a^3, a\tilde{z}_2'$.

7 Final Comments and Remarks

7.1 Relations to Non-Abelian Gauge Theories and Sigma Models

As we mentioned, this article is a companion Reference with further detailed calculations supporting other shorter articles [18–20]. Now we make final comments and remarks on how our cobordism group calculations in the preceding Sec. 5 and Sec. 6 are applied in these works [18–20].

1. *Pure SU(2) Yang-Mills theory's higher anomaly*: Ref. [9] introduces the generalized global symmetries include higher symmetries (See a brief review in Sec. 1.4, Items (◇ 1) and (◇ 2)). The pure SU(N) Yang-Mills (YM) gauge theory has a higher-1-dimensional (1-form) electric symmetry, denoted as $\mathbb{Z}_{\mathrm{N},[1]}^e$ (previously known as the $\mathbb{Z}_\mathrm{N}$-center symmetry). The pure SU(N) YM theory in 4d has the corresponding 1-form electric $\mathbb{Z}_{\mathrm{N},[1]}^e$ symmetry charged object: the 1-dimensional gauge-invariant Wilson line W_e:

$$W_e = \mathrm{Tr}_\mathrm{R}(\mathrm{P}\exp(\mathrm{i} \oint a)). \tag{7.1}$$

and the 2-dimensional charge operator: the 2-dimensional charge surface operator U_e.

$$U_e = \exp(\mathrm{i}\frac{2\pi}{\mathrm{N}} \oint \Lambda), \tag{7.2}$$

The spacetime path integral formulation of SU(N) YM higher symmetry becomes a relation:

$$\langle W_e\, U_e\rangle = \langle \mathrm{Tr}_\mathrm{R}(\mathrm{P}\exp(\mathrm{i} \oint_{\gamma^1} a))\ \exp(\mathrm{i}\pi \oint_{\Sigma^2} \Lambda)\rangle = \exp(\frac{\mathrm{i}2\pi}{\mathrm{N}}\mathrm{Lk}(\gamma^1, \Sigma^2)), \tag{7.3}$$

with R in fundamental representation. The remarkable Ref. [16] discovers the mixed higher 't Hooft anomaly of pure SU(N) YM theory at an even integer N with a second Chern class topological term ($\pi \int_{M^4} c_2 \simeq \int_{M^4} \frac{\theta}{8\pi^2}\mathrm{Tr}\, F_a \wedge F_a$ at $\theta = \pi$ with the YM field strength curvature F_a) between time-reversal $\mathbb{Z}_2^T$ symmetry (with a schematic background field $\mathcal{T}$ or $w_1(TM)$) and the 1-form electric $\mathbb{Z}_{\mathrm{N},[1]}^e$ symmetry (with a schematic 2-form background field B), via a schematic 5d topological term:

$$\sim \exp(\mathrm{i}\pi \int_{M^5} \mathcal{T}BB). \tag{7.4}$$

Ref. [18, 20] shows that this precise 5d topological term written as a 5d bordism invariant (at $\mathrm{N}=2$) of a mod 2 class term is:

$$\exp(\mathrm{i}\pi \int_{M^5} B\mathrm{Sq}^1 B + \mathrm{Sq}^2\mathrm{Sq}^1 B) = \exp(\mathrm{i}\frac{\pi}{2}\int_{M^5} \tilde{w}_1(TM)\cup \mathcal{P}(B)), \tag{7.5}$$

based on the notation introduced earlier in Sec. 5.4.1 and Sec. 5.4.1, it can be written as:

$$\exp(\mathrm{i}\pi \int_{M^5} x_2\mathrm{Sq}^1 x_2 + \mathrm{Sq}^2\mathrm{Sq}^1 x_2) = \exp(\mathrm{i}\pi \int_{M^5} x_2x_3 + x_5) = \exp(\mathrm{i}\pi \int_{M^5} \frac{1}{2}\tilde{w}_1\mathcal{P}_2(x_2)), \tag{7.6}$$

where $x_2 = B$ is the generator of $\mathrm{H}^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)$. Other than Ref. [18, 20], the derivation of the relation of the topological invariant $x_2x_3 + x_5 = \frac{1}{2}\tilde{w}_1\mathcal{P}_2(x_2)$ has also been examined in an excellent note of Debray [69]. For eqn. (7.6), our relevant cobordism theory includes the unoriented bordism group $\Omega_d^{\mathrm{O}}(\mathrm{B}^2\mathbb{Z}_2)$ in Sec. 5.4.1 and the oriented bordism group $\Omega_d^{\mathrm{SO}}(\mathrm{B}^2\mathbb{Z}_2)$ in Sec. 5.4.2.

2. *Pure SU(N) Yang-Mills theory's higher anomaly*: The above formulas (7.5) and (7.6), are 5d topological invariants characterizing the 4d SU(2) YM at $\theta = \pi$'s higher anomaly. For a generic 4d SU(N) YM at $\theta = \pi$ of even integer $\mathrm{N} = 2^n$, Ref. [18] proposes a precise 5d topological term written as a 5d bordism invariant (at $\mathrm{N} = 2^n$) which includes at least a mod 2 class term:

$$B\beta_{(2,\mathrm{N}=2^n)}B + \frac{\mathrm{N}}{2}\mathrm{Sq}^2\beta_{(2,\mathrm{N})}B = \frac{1}{\mathrm{N}}\tilde{w}_1(TM)\mathcal{P}(B), \tag{7.7}$$

   characterizing (part of) the 4d SU(2) YM at $\theta = \pi$'s higher anomaly. Pontryagin square is defined as $\mathcal{P} : \mathrm{H}^2(-, \mathbb{Z}_{2^n}) \to \mathrm{H}^4(-, \mathbb{Z}_{2^{n+1}})$. For example, at $\mathrm{N} = 2$, we get eqn. (7.7) coincides the same formula as eqn. (7.6). At $\mathrm{N} = 4$, we get the formula $B\beta_{(2,\mathrm{N}=4)}B = \frac{1}{4}\tilde{w}_1(TM)\mathcal{P}(B)$. Our corresponding cobordism group calculations are presented in Sec. 6.2.1 and Sec. 6.2.2.

3. *More discrete symmetries (e.g. charge conjugation) and more higher anomalies*: SU(N) YM theory has charge conjugation symmetry $\mathbb{Z}_2^C$ when $\mathrm{N} > 2$. Therefore, Ref. [18] presents additional higher 't Hooft anomalies associated to the charge conjugation $\mathbb{Z}_2^C$ background field A_C. The relevant cobordism group calculation involving additional $\mathbb{Z}_2^C$ symmetry requires adding a new $\mathrm{B}\mathbb{Z}_2$ sector into the previous classifying space. Relevant cobordism group calculations are presented in Ref. [18], and also some trial toy-model examples in Sec. 5.6, Sec. 6.6, and Sec. 6.7 involving the classifying space $\mathrm{B}\mathbb{Z}_m$ and higher-classifying space $\mathrm{B}^2\mathbb{Z}_n$. The combined higher-classifying space includes the forms of $\mathrm{B}\mathbb{Z}_m \times \mathrm{B}^2\mathbb{Z}_n$ or $\mathrm{B}\mathbb{Z}_m \ltimes \mathrm{B}^2\mathbb{Z}_n$ (in Ref. [18]).

4. *Non-linear sigma models and their anomalies*: Non-linear sigma models such as the $\mathbb{CP}^{\mathrm{N}-1}$-sigma models (with the target space $\mathbb{CP}^{\mathrm{N}-1}$) have a global symmetry of PSU(N). Therefore, the relevant cobordism group calculations presented in Ref. [18] include the classifying space BPSU(N). We include the pertinent cobordism group calculations also for BPSU(N) in Sec. 5.5, BPSU(2)=BO(3) in Sec. 6.3, and $\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(N))$ in Sec. 6.8. The time reversal symmetry $\mathbb{Z}_2^T$ of bosonic or fermionic version of sigma models corresponds to O or $\mathrm{Pin}^{\pm}$ structure respectively. The charge conjugation symmetry $\mathbb{Z}_2^C$ corresponds to the $\mathrm{B}\mathbb{Z}_2$ in $\mathrm{B}(\mathbb{Z}_2 \ltimes \mathrm{PSU}(N))$ in Sec. 6.8

5. *Higher-symmetry extension, and the fate of gapped and gapless-ness of quantum phases*:
   An SU(N) YM gauge theory coupled to SU(N) fundamental fermions break explicitly the 1-form $\mathbb{Z}^e_{\mathrm{N},[1]}$-symmetry (thus does not have the 1-form $\mathbb{Z}^e_{\mathrm{N},[1]}$-symmetry). An SU(N) YM gauge theory coupled to SU(N) adjoint fermions can still possess a 1-form $\mathbb{Z}^e_{\mathrm{N},[1]}$-symmetry.

   The SU(N) adjoint fermion YM gauge theory is known as the adjoint QCD of SU(N) gauge group. The relevant global symmetries of this adjoint QCD thus includes $\mathbb{Z}^e_{\mathrm{N},[1]}$ and a

SU(m) flavor chiral symmetry (say, if there is an m-flavor of Wely fermions in the adjoint representation of SU(N)). Some trial toy-model examples of cobordism groups, involving these classifying spaces BSU(m), BPSU(m) and $\mathrm{B}^2\mathbb{Z}_\mathrm{N}$, are presented in Sec. 5.7. For example, for the adjoint QCD with an SU(2) gauge group and $\mathrm{N}_f = 2$ adjoint Weyl fermions, the pertinent symmetry groups are $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right) \times \mathbb{Z}_{2,[1]}^e$ or $\mathrm{Pin}^- \times_{\mathbb{Z}_2^F} \left(\frac{\mathrm{SU}(2)\times\mathbb{Z}_{8,\mathrm{A}}}{\mathbb{Z}_2^F}\right) \times \mathbb{Z}_{2,[1]}^e$ (including a time-reversal symmetry), see their cobordism groups and higher-anomalies in [18].

Along this development, the fate of relevant theories of the adjoint QCD is explored recently using the modern language of higher-symmetries and higher-anomalies in various other Ref. [17, 70–74], other than [18], and References therein.

Ref. [18] employs a generalization of a symmetry-extension method of [43] to a higher-symmetry-extension method, as a tool of constructing a fully-symmetry-preserving gapped phase saturating the higher 't Hooft anomalies. It turns out that:

- Certain higher 't Hooft anomalies *cannot* be saturated by a fully-symmetry-preserving gapped phase (e.g. TQFT); which implies either the symmetry-breaking or gapless-ness of the dynamical fate of the theories. Examples include $\mathcal{P}(B)$ in $\mathrm{H}^4(M, \mathbb{Z}_4)$ and $A\mathcal{P}(B)$ in $\mathrm{H}^5(M, \mathbb{Z}_4)$ where M is the spacetime manifold [18]. This higher-symmetry-extension approach [18] thus rules out some candidate low-energy infrared phases (as a dual phase of a high-energy QFT) proposed in [71].
- Certain higher 't Hooft anomalies *can* be saturated by a fully-symmetry-preserving gapped phase (e.g. TQFT); which implies a possible exotic dynamical fate as *the confinement with no chiral symmetry nor 1-form center symmetry breaking*. Various examples of pure SU(N) YM gauge theories with $\theta = \pi$-topological term indeed afford such an exotic confinement without any (ordinary or higher) symmetry-breaking, see Ref. [18, 20].

7.2 Relations to Bosonic/Fermionic Higher-Symmetry-Protected Topological states: Beyond Generalized Super-Group Cohomology Theories

1. *Bosonic higher-symmetry protected topological states* (b-higher-SPTs):

   (1) Bosonic symmetry-protected topological states (bSPTs) in $d+1$d of an internal (ordinary 0-form) global symmetry $G_{(0)}$ is proposed firstly in Chen-Gu-Liu-Wen Ref. [5] to be classified by a cohomology group
$$\mathrm{H}^{d+1}(G_{(0)}, \mathrm{U}(1)) \tag{7.8}$$
   or the topological cohomology of classifying space $\mathrm{B}G_0$ as
$$\mathcal{H}^{d+1}(\mathrm{B}G_{(0)}, \mathrm{U}(1)). \tag{7.9}$$
   (2) It is later proposed by Kapustin in Ref. [26], for bosonic SPTs of $G_{(0)}$ and for bosonic symmetric invertible topological order (denoted as b-iTO) of $G_{(0)}$, they are classify by a cobordism group classification, which is beyond the group cohomology framework. The torsion (finite group $\prod_j \mathbb{Z}_{n_j}$) part of cobordism group classification contains:
$$\mathrm{Hom}(\Omega_{d+1,\mathrm{tors}}^H(\mathrm{B}G_{(0)}), \mathrm{U}(1)) \tag{7.10}$$
   where H is an oriented $H = \mathrm{SO}$ or an unoriented $H = \mathrm{O}$ for the (co)bordism group. To include the free part (the non-torsion part, infinite integer $\prod_j \mathbb{Z}$ classes), we need to include additional

contribution: In physics, this is related to the nontrivial thermal Hall response and gravitational Chern-Simons terms.

(3) Ref. [53] of Wen proposes the $\mathrm{SO}(\infty)$ version of bosonic cohomology group to classify the bSPTs beyond Chen-Gu-Liu-Wen's Ref. [5] via

$$\mathrm{H}^{d+1}(\mathrm{SO}(\infty) \times G_{(0)}, \mathrm{U}(1)), \quad \mathcal{H}^{d+1}(\mathrm{B}(\mathrm{SO}(\infty) \times G_{(0)}), \mathrm{U}(1)). \tag{7.11}$$

(4) Ref. [4] of Freed-Hopkins introduces this classification of topological phases (TP), including the torsion and the free parts, defined as a suitable new cobordism group denoted:

$$\mathrm{TP}_{d+1}(H \times G_{(0)}). \tag{7.12}$$

(5) In our work, we generalize the result of Ref. [4] of bosonic SPTs to bosonic higher-SPTs including the higher-symmetries (e.g. $G_{(1)}$), such as

$$\mathrm{TP}_{d+1}(H \times (G_{(0)} \times \mathrm{B}G_{(1)})), \quad \mathrm{TP}_{d+1}(H \times (G_{(0)} \ltimes \mathrm{B}G_{(1)})), \quad \dots \tag{7.13}$$

and more general constructions in Sec. 4.

2. *Fermionic higher-symmetry protected topological states* (f-higher-SPTs):
Fermionic symmetry-protected topological states (fSPTs) in $d+1$d of an internal (ordinary 0-form) global symmetry $G_{(0)}$ is proposed in Ref. [75] by Gu-Wen to be classified by a super-cohomology group. A corrected modification of Gu-Wen model is presented by Gaiotto-Kapustin in Ref. [76]. The Gu-Wen model and Gaiotto-Kapustin model is more or less complete for the 3d (2+1D) fSPTs with a global symmetry of finite group $G_{(0)}$. They also provide lattice Hamiltonian or wavefunction model constructions. The relation between the full fermionic symmetry group G_F and $G_{(0)}$ is based on a short exact sequence, extended by a normal subgroup fermionic parity $\mathbb{Z}_2^F$:

$$1 \to \mathbb{Z}_2^F \to G_F \to G_{(0)} \to 1. \tag{7.14}$$

However, neither Gu-Wen nor Gaiotto-Kapustin models obtain a complete classification for 4d (3+1D) fSPTs, even for a global symmetry of finite group $G_{(0)}$. Improvements are made via several different approaches:

(1) Kitaev's in Ref. [77] proposes a homotopy-theoretic approach to SPT phases in action. This gives rise a correct $\mathbb{Z}_{16}$ classification of 3+1D topological superconductors, matching to the cobordism group classification. Kitaev's Ref. [77] can be regarded as the interaction version of SPT classification, improved from his previous K-theory approach for the topological phase classification of free-fermion systems [78]. Kitaev's approach is reviewed, for example, in Ref. [79, 80].

(2) Kapustin-Thorngren-Turzillo-Wang [27] approaches is based on the $H = \mathrm{Spin}$ or $\mathrm{Pin}^{\pm}$ versions of cobordism group $\mathrm{Hom}(\Omega^{H}_{d+1,\mathrm{tors}}(\mathrm{B}G_{(0)}), \mathrm{U}(1))$.

(3) Freed-Hopkins [4] introduces a cobordism group $\mathrm{TP}_{d+1}(H \times G_{(0)})$ whose effective computation is based on the Adams spectral sequence, with $H = \mathrm{Spin}$ or $\mathrm{Pin}^{\pm}$ for a fermionic theory.

(4) Kapustin-Thorngren in Ref. [37] introduces the higher-dimensional bosonization to construct higher-dimensional fSPTs, mostly focusing on a finite symmetry group G_F.

(5) Wang-Gu in Ref. [81, 82] introduces a generalized group super-cohomology theory with

multi-layers of group extension structures of super-cohomology group, mostly focusing on a finite symmetry group G_F. The computation of fSPTs classification based on the generalized super-cohomology group is similar to the Atiyah-Hirzebruch spectral sequence method. See related discussions in Ref. [10,83] on Atiyah-Hirzebruch spectral sequence for classifying fSPTs.

(6) Ref. [40, 84] use a mixture of Dai-Freed theorem [50] and Atiyah-Hirzebruch-like spectral sequence to determine fSPTs and their discrete anomalies on the boundaries.

(7) Ref. [10] computes various finite-group fSPTs via Adams spectral sequence. Their methods and their derived fSPT terms can be regarded as the complementary approach to those derived via Atiyah-Hirzebruch-like spectral sequence [81–83].

(8) In our work, we generalize the result of Ref. [4] of fermionic SPTs to fermionic higher-SPTs including the higher-symmetries (e.g. $G_{(1)}$), such as

$$\mathrm{TP}_{d+1}(H \times (G_{(0)} \times \mathrm{B}G_{(1)})), \quad \mathrm{TP}_{d+1}(H \times (G_{(0)} \ltimes \mathrm{B}G_{(1)})), \quad \dots$$

with H = Spin or $\mathrm{Pin}^{\pm}$. Or slightly more generally, consider the classification of fermionic higher-SPTs via:

$$\mathrm{TP}_{d+1}(\mathbb{H})$$

such that the H, $\mathbb{G}$ and $\mathbb{H}$ satisfy the following exact sequences:

$$\begin{cases} 1 \to \mathbb{G} \to \mathbb{H} \to \mathrm{Spin}(d+1) \to 1, \\ 1 \to \mathbb{Z}_2^F \to \mathrm{Spin}(d+1) \to \mathrm{SO}(d+1) \to 1, \\ \mathrm{B}^2 G_{(1)} \to \mathrm{B}\mathbb{G} \to \mathrm{B}G_{(0)} \to \mathrm{B}^3 G_{(1)} \to \dots. \end{cases} \tag{7.15}$$

or

$$\begin{cases} 1 \to \mathbb{G} \to \mathbb{H} \to \mathrm{Pin}^{\pm}(d+1) \to 1, \\ 1 \to \mathbb{Z}_2^F \to \mathrm{Pin}^{\pm}(d+1) \to \mathrm{O}(d+1) \to 1, \\ \mathrm{B}^2 G_{(1)} \to \mathrm{B}\mathbb{G} \to \mathrm{B}G_{(0)} \to \mathrm{B}^3 G_{(1)} \to \dots. \end{cases} \tag{7.16}$$

Even more general constructions are explored in Sec. 4.

3. *Braiding statistics and link invariants approach to characterize bosonic/fermionic SPTs and higher-SPTs*: Another useful approach to classify SPTs is based on gauging the global symmetry group of SPTs, such that we obtain a gauge theory or TQFT at the end. The braiding statistics of the fractionalized excitations of gauged SPTs can characterize the pre-gauged SPTs, the explicit method of 3d (2+1D) SPTs is outlined by Levin-Gu [85]. Here we focus on the case of continuum field theory formulation of braiding statistics and link invariants approach to characterize these higher-dimensional SPTs.

(1) 4d (3+1D) bSPTs: Ref. [86, 87]
(2) 4d (3+1D) fSPTs: Ref. [10, 37, 87, 88]
(3) 5d (4+1D) SPTs or higher dimensions: Ref. [20].

4. *A Generalized Cobordism Theory of higher-symmetry groups — beyond Higher-Group Super-Cohomology Theories*:

It is known that the cobordism theory approach of Kapustin et al. [26, 27] and Freed-Hopkins [4] obtain the classification of fSPTs and bSPTs beyond Chen-Gu-Liu-Wen's group cohomology [5] or Gu-Wen's group super-cohomology [75]. A more refined version of generalized group super-cohomology [81, 82] can obtain some missing classes of [75] to match the cobordism classification.

Therefore, we expect that the our approach, on a generalized cobordism theory including the higher-symmetry groups, can classify higher-SPTs (including fSPTs and bSPTs) that may or may not be captured by higher-group super-cohomology theories.

For future work, it will be illuminating to understand the distinctions between the generalized higher-group cobordism theory approach and the generalized higher-group super-cohomology theories. We expect the comparison between two approaches can be rephrased as a certain version of Adams spectral sequence method in contrast to a certain version of Atiyah-Hirzebruch spectral sequence method.

It will also be important to figure the possible lattice-regularization (e.g. lattice Hamiltonian on simplicial complex) of those higher-SPTs classified by our generalized cobordism theory.

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